Geometric Representation of Space-Time

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Spacetime:

Spacetime is any mathematical model that fuses the three dimensions of space and the one dimension of time into a single four-dimensional continuum (a continuous sequence in which adjacent elements are not perceptibly different from each other). Spacetime diagrams can be used to visualize relativistic effects such as why different observers perceive *where* and *when* events occur.

Minkowski Space:

Minkowski space (or Minkowski spacetime) is a combination of three-dimensional Euclidean space and time into a four-dimensional manifold where the spacetime interval between any two events is independent of the inertial frame of reference in which they are recorded. Although initially developed by mathematician Hermann Minkowski for Maxwell's equations of electromagnetism, the mathematical structure of Minkowski spacetime was shown to be an immediate consequence of the postulates of special relativity.

Minkowski space is closely associated with Einstein's theory of special relativity, and is the most common mathematical structure on which special relativity is formulated. While the individual components in Euclidean space and time may differ due to length contraction and time dilation, in Minkowski spacetime, all frame of references will agree on the total distance in spacetime between events. Because it treats time differently than it treats the 3 spatial dimensions, Minkowski space differs from four-dimensional Euclidean space.

Space-Time Diagrams:

According to classical physics, the time coordinate is unaffected by a transformation from one inertial frame to another i.e. the time coordinate, t', of one inertial system does not depend on the space coordinates, x, y, z of another inertial system, the transformation equation being t' = t.

In relativity, however, space and time are interdependent. The time coordinate of one inertial system depends on both the time and the space coordinates of another inertial system, the transformation equation being

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1}$$

Hence, instead of treating space and time separately, as is quite properly done in classical theory, it is natural in relativity to treat them together. H. Minkowski was first to show clearly how this could be done. In what follows, we shall consider only one space axis, the x-axis, and shall ignore the y and z axes. We lose no generality by this algebraic simplification and this procedure will enable us to focus more clearly on the interdependence of space and time and its geometric representation. The coordinates of an event are given then by x and t. All possible space-time coordinates can be represented on a space-time diagram in which the space axis is horizontal and the time axis is vertical. It is convenient to keep the dimensions of the coordinates the same; this is easily done by multiplying the time t by the universal constant c, the velocity of light. Let ct be represented by the symbol w. Then, the Lorentz transformation equations can be written as follows:

$$x' = \frac{x - \beta w}{\sqrt{1-\beta^2}} \qquad x = \frac{x' + \beta w'}{\sqrt{1-\beta^2}} \tag{2a}$$

$$w' = \frac{w - \beta x}{\sqrt{1-\beta^2}} \qquad w = \frac{w' + \beta x'}{\sqrt{1-\beta^2}} \tag{2b}$$

Notice the symmetry in this form of the equations. To represent the situation geometrically, we begin by drawing the x and w axes of frame S orthogonal (perpendicular) to one another (Fig. 1).

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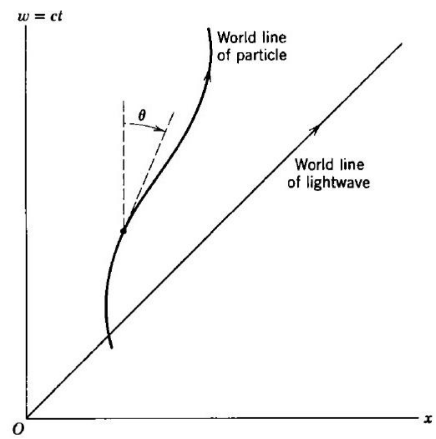

Fig. 1.

If we wanted to represent the motion of a particle in this frame, we would draw a curve, called a **world line**, which gives the loci of space-time points corresponding to the motion. Minkowski referred to space-time as "**the world**." Hence, events are **world points** and a collection of events giving the history of a particle is a world line. Physical laws on the interaction of particles can be thought of as the geometric relations between their world lines. In this sense, Minkowski may be said to have geometrized physics.

The tangent to the world line at any point, being $\frac{dx}{dw} = \frac{1}{c}\frac{dx}{dt}$, is always inclined at an angle less than 45° with the time axis. For this angle (see Fig. 1) is given by $\tan\theta = \frac{dx}{dw} = \frac{u}{c}$ and we must have u < c for a material particle. The world line of a light wave, for which u = c, is a straight line making a 45° angle with the axes.

Spacetime Interval:

Under the proper conditions, different observers will disagree on the length of time between two *events* (because of time dilation) or the distance between the two events (because of length contraction). So, the space and time are separately not invariant that's why they can be merged into a four dimensional spacetime continuum. Special relativity provides a new invariant, called the ***spacetime interval***, which combines distances in space and in time. All observers who measure time and distance carefully will find the same ***spacetime interval*** between any two events. Suppose an observer measures two events as being separated in time by Δt and a spatial distance Δx. Then the spacetime interval ΔS between the two events that are separated by a distance Δx in space and by Δct = cΔt in the ct-coordinate is:

$$(\Delta S)^2 = (\Delta ct)^2 - (\Delta x)^2$$

for three space dimensions, $(\Delta S)^2 = (\Delta ct)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$

It can also be written as, $(\Delta S)^2 = -(\Delta ct)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$

The constant c, the speed of light is a conversion factor to make both axis in the same dimension of length.

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Geometric Interpretation of Lorentz Transformation

Consider now the primed frame (S') which moves relative to S with a velocity v along the common x-x' axis. The equation of motion of S' relative to S can be obtained by setting x' = 0 (which locates the origin of S'); from equation 2a, we see that this corresponds to $x = \beta w$ (= vt). We draw the line x' = 0 (that is, $x = \beta w$) on our diagram (Fig. 2) and note that, since v < c and $\beta < 1$, the angle which this line makes with the w-axis, ϕ= tan-1 (β), is less than 45°. Just as the w-axis corresponds to x = 0 and is the time axis in frame S, so the line x' = 0 gives the time axis w' in S'.

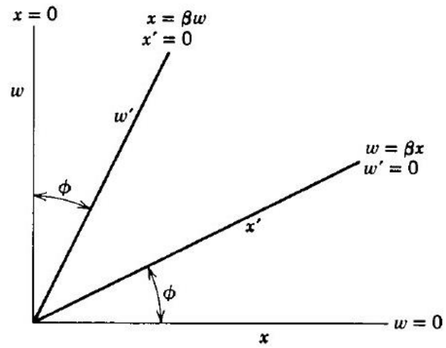

Fig. 2.

Now, if we draw the line w' = 0 (giving the location of clocks which read t' = 0 in S'), we shall have the space axis x'. That is, just as the x-axis corresponds to w = 0, so the x'-axis corresponds to w' = 0. But, from Eq. 2b, w' = 0 gives us $w = \beta x$ as the equation of this axis on our w-x diagram (Fig. 2). The angle between the space axes is the same as that between the time axes.

From Fig. 2, we see that in four-space (x.y.z.t) the Lorentz equations involve transforming from an orthogonal system to a non-orthogonal system. We can use this representation to show the relativity of simultaneity and to give a geometrical interpretation of the space-contraction and time-dilation effects, as well as to illustrate their reciprocal nature. To do all this clearly, let us first represent the situation on a new diagram (Fig. 3). Here we draw the two branches of the hyperbola $w^2 - x^2 = 1$, and the two branches of the hyperbola $x^2 - w^2 = 1$. These lines whose meaning will soon be clear, approach asymptotically the 45° light ray world-lines. We also draw in the x, w axes of S and the x', w' axes of S'.

The space-time point P_1 is the intersection of the right branch of hyperbola $x^2 - w^2 = 1$ with the x'-axis given by $w = \beta x$. Hence, P_1 is on both these lines and its coordinates (obtained by combining the equations of the lines) are

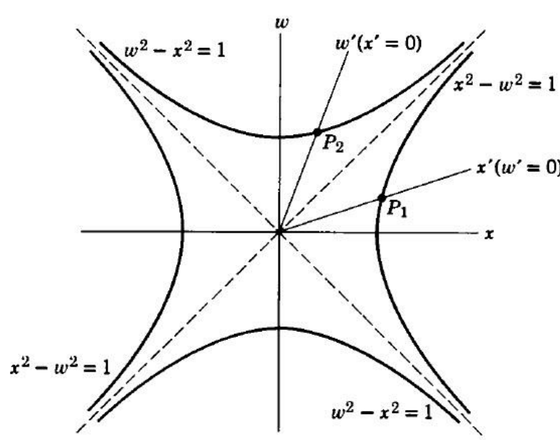

Fig. 3

$$w = \frac{\beta}{\sqrt{1-\beta^2}} \quad \text{and} \quad x = \frac{1}{\sqrt{1-\beta^2}} \tag{3a}$$

But, comparison of Eq. 3a with Eq. 2 shows that Eq. 3a represents unit length (i.e., x' = 1) and zero time (i.e., w' = 0) in the S' frame. That is, the interval OP_1 gives unit length along the x'-axis. Similarly, the space-time point P_2 is the intersection of the upper branch of hyper-bola $w^2 - x^2 = 1$ with the w'-axis given by $x = \beta w$. Hence, P_2 is on both these lines and its coordinates (obtained by combining the equations of the lines) are

$$x = \frac{\beta}{\sqrt{1-\beta^2}} \quad \text{and} \quad w = \frac{1}{\sqrt{1-\beta^2}} \tag{3b}$$

Comparison of Eq. 3b with Eq. 2 shows that Eq. 3b represents unit time (i.e., w' = 1) and zero length (i.e., x' = 0) in the S' frame. That is, the interval OP_2 gives unit time along the w'-axis. The hyperbolas are often referred to as calibration curves. Consider the upper hyperbola, for example. At x = 0, we have w = 1, which (in units of ct) is unit time in S. At any other point x we have $c^2t^2 - x^2 = c^2(t^2 - x^2/c^2) = C^2\tau^2 = 1$. Thus, points on the upper hyperbola give unit time on the clock at rest in S'; that is, the proper time in units of ct is equal to one. Whatever the relative velocity of S' to S, the intersection of the time axis with this hyperbola will give the unit time in S'. Similarly, for the right hyperbola we have x = 1 at w = 0, which is unit length in S (measured from the origin).

At any other value of w, points on the hyperbola represent unit length at rest in a frame S', the velocity of S' relative to S being determined by the inclination of the space axis, which intersects the hyperbola at the point in question. Let us suppose now that we observe events from two inertial frames, S and S', whose relative velocity we know. The hyperbolic calibration curves determine the unit time interval and unit length interval on the axes of these frames; once the hyperbolas have served this purpose, we can dispense with them. In Fig. 4 we show the calibration of the axes S and S', the unit time interval along w' being a longer line segment than the unit time interval along w and the unit length interval along x' being a longer line segment than the unit length interval along x. The first thing we must be able to do is to determine the space-time coordinates of an event, such as P, from the Minkowski diagram. To find the space coordinate of the event, we simply draw a line parallel to the time axis from P to the space axis. The time coordinate is given similarly by a line parallel to the space axis from P to the time axis. The rules hold equally well for the primed frame as for the unprimed frame. In Fig. 4, for example, the event P has space-time coordinates x = 3, w = 2.5 in S (dashed lines), and space-time coordinates x = 2, w' = 1.5 in S' (dotted lines). It is as though the rectangular grid of coordinate lines of S (Fig. 5a) become squashed toward the bisecting 45° line when the coordinate lines of S' are put on the same graph (Fig. 5b); clearly the Lorentz equations transform an orthogonal system to a non-orthogonal one.

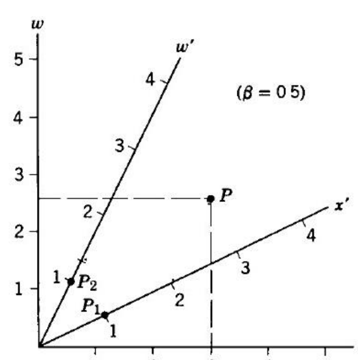

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Fig. 4

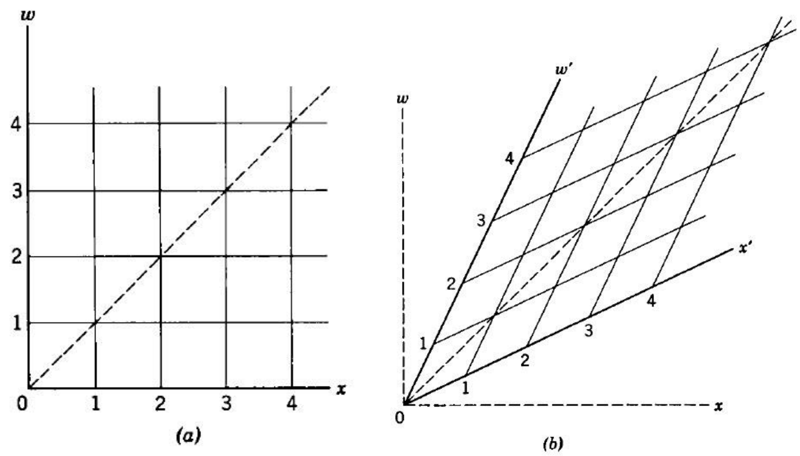

Fig. 5

Simultaneity, Contraction, and Dilation

Now we can easily show the relativity of **simultaneity**. As measured in S', two events will be simultaneous if they have the same time coordinate w'. Hence, if the events lie on a line parallel to the x'-axis they are simultaneous to S'. In Fig. 6, for example, events Q_1 and Q_2 are simultaneous in S'; they obviously are not simultaneous in S, occurring at different times w_1 and w_2 there. Similarly, two events R_1 and R_2, which are simultaneous in S, are separated in time in S'.

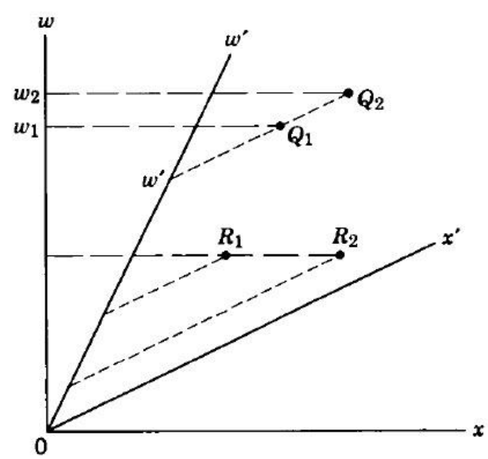

Fig. 6

As for the **space contraction**, consider Fig. 7a. Let a meter stick be at rest in the S-frame, its end points being at x = 3 and x = 4, for ex-ample. As time goes on, the world-line of each end point traces out a vertical line parallel to the w-axis. The length of the stick is defined as the distance between the end points measured simultaneously. In S, the rest frame, the length is the distance in S between the intersections of the world lines with the x-axis, or any line parallel to the x-axis, for these intersecting points represent simultaneous events in S. The rest length is one meter. To get the length of the stick in S', where the stick moves, we must obtain the distance in S' between end points measured simultaneously. This will be the separation in S' of the intersections of the world lines with the x'-axis, or any line parallel to the x'-axis, for these intersecting points represent simultaneous events in S'. The length of the (moving) stick is clearly less than one meter in S'. (See Fig. 7a). Notice how very clearly Fig. 7a reveals that it is a disagreement about the simultaneity of events that leads to different measured lengths. Indeed, the two observers do not measure the same pair of events in determining the length of a body (e.g., the S-observer uses E_1 and E_2, say, whereas the S'-observer would use E_1 and E_3, or E_2 and E_4) for events which are simultaneous to one inertial observer are not simultaneous to the other (see Ref. 2 for a forceful presentation of this point). We should also note that the x' coordinate of each endpoint decreases as time goes on (simply project from successive world-line points parallel to w' onto the x'-axis), consistent with the fact that the stick which is at rest in S moves towards the left in S'.

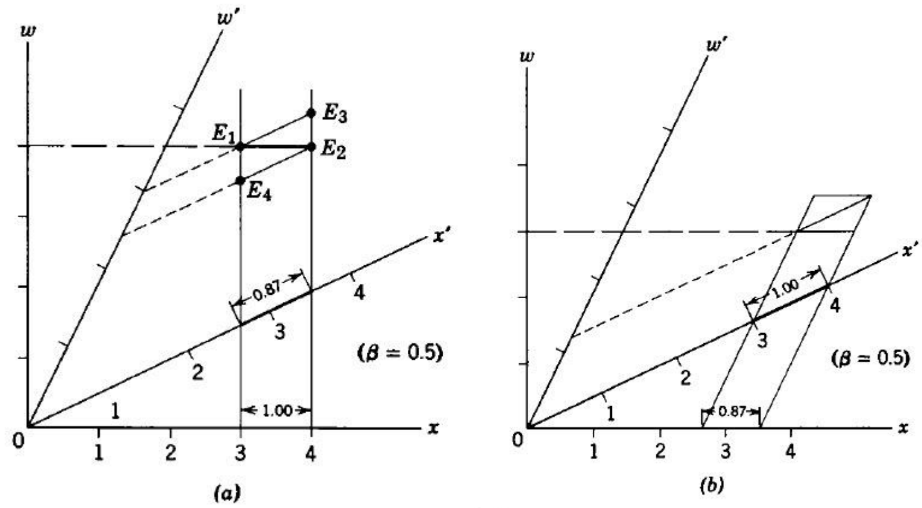

Fig. 7

The reciprocal nature of this result is shown in Fig. 7b. Here, we have a meter stick at rest in S' and the world lines of its end points are parallel to w' (the end points are always at x' = 3 and x' = 4, say). The rest length is one meter. In S, where the stick moves to the right, the measured length is the distance in S between intersections of these world lines with the x-axis, or any line parallel to the x-axis. The length of the (moving) stick is clearly less than one meter in S (see Fig. 7b).

It remains now to demonstrate the **time-dilation** result geometrically. For this purpose consider Fig. 8. Let a clock be at rest in frame S, ticking off units of time there. The solid vertical line in Fig. 8, at x = 2.3, is the world line corresponding to such a single clock. T_1 and T_2 are the events of ticking at w (=ct) = 2 and w (= ct) = 3, the time interval in S between ticks being unity. In S', this clock is moving to the left so that it is at a different place there each time it ticks. To measure the time interval between events T_1 and T_2 in S', we use two different clocks, one at the location of event T_1 and the other at the location of event T_2. The difference in reading of these clocks in S' is the difference in times between T_1 and T_2 as measured in S'. From the graph, we see that this interval is greater than unity. Hence, from the point of view of S', the moving S-clock appears slowed down. During the interval that the S-clock registered unit time, the S'-clock registered a time greater than one unit. The reciprocal nature of the time-dilation result is also shown in Fig. 8.

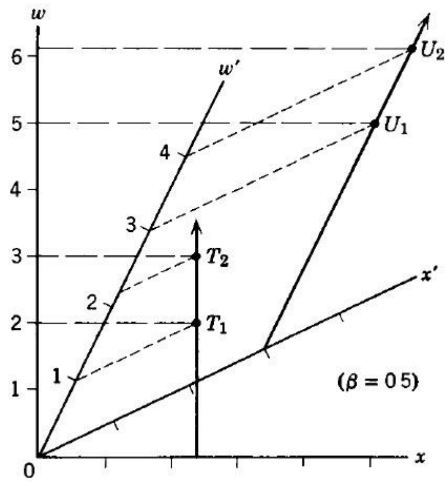

Fig. 8

The student should construct the detailed argument. Here a clock at rest in S' emits ticks U_1 and U_2 separated by unit proper time. As measured in S, the corresponding time interval exceeds one unit.

The Time Order of Events

We can also use the geometrical representation of space-time to gain further insight into the concepts of simultaneity and the time order of events. Consider the shaded area in Fig. 9, for example. Through any point P in this shaded area, bounded by the world lines of light waves, we can draw a w'-axis from the origin; that is, we can find an inertial frame S' in which the events O and P occur at the same place (x' = 0) and are separated only in time. As shown in Fig. 9, event P follows event O in time (it comes later on S' clocks), as is true wherever event P is in the upper half of the shaded area. Hence, events in the upper half (region 1 on Fig. 10) are absolutely in the future relative to 0 and this region is called the Absolute Future. If event P is at a space-time point in the lower half of the shaded area (region 2 on Fig. 10) then P will precede event O in time. Events in the lower half are absolutely in the past relative to O and this region is called the Absolute Past. In the shaded regions, therefore, there is a definite time order of events relative to O for we can always find a frame in which O and P occur at the same place; a single clock will determine absolutely the time order of the event at this place.

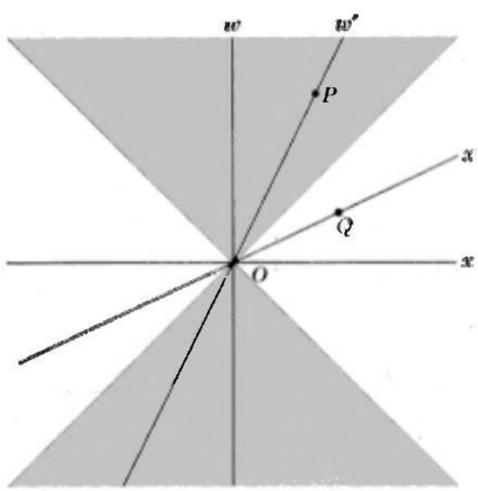

Fig. 9

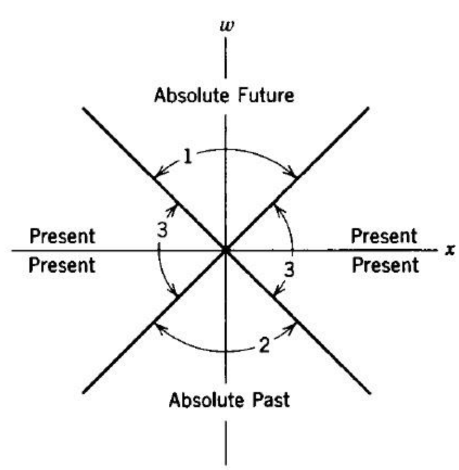

Fig. 10

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Space Separation of Events

Consider now the unshaded regions of Fig. 9. Through any point Q we can draw an x'-axis from the origin; that is, we can find an inertial frame S' in which the events O and Q occur at the same time (w' = ct' = 0) and are separated only in space. We can always find an inertial frame in which events O and Q appear to be simultaneous for space-time points Q that are in the unshaded regions (region 3 of Fig. 10), so that this region is called the Present. In other inertial frames, of course, O and Q are not simultaneous and there is no absolute time order of these events but a relative time order, instead.

If we ask about the space separation of events, rather than their time order, we see that events in the present are absolutely separated from O, whereas those in the absolute future or absolute past have no definite space order relative to O. Indeed, region 3 (present) is said to be "space-like" whereas regions 1 and 2 (absolute past or future) are said to be "timelike." That is, a world interval such as OQ is spacelike and a world interval such as OP is timelike. The geometrical considerations that we have presented are connected with the invariant nature of proper time, that is, with the relation $d\tau^2 = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)$. We can illustrate as following

$$(\tau_2 - \tau_1)^2 = (t_2 - t_1)^2 - \frac{1}{c^2}\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]$$

let subscript one refer to the origin ($t_1 = 0 = x_1 = y_1 = z_1$) and let subscript two refer to any other space-time point, so that

$$\tau^2 = t^2 - \frac{1}{c^2}(x^2 + y^2 + z^2)$$

Now, in our case, we have ignored y and z so that the appropriate expression is $\tau^2 = t^2 - \frac{x^2}{c^2}$. We can write this conveniently as $c^2\tau^2 = c^2t^2 - x^2$ which, in our terminology, is simply $c^2\tau^2 = w^2 - x^2$. The quantity $c^2\tau^2$ is an invariant, that is, $w^2 - x^2 = w'^2 - x'^2$ for the same two events. Hence, the quantity $-c^2\tau^2$, which we shall call σ^2, is also invariant. We have then the two relations:

$$c^2\tau^2 = w^2 - x^2$$

$$\sigma^2 = x^2 - w^2$$

Now consider Figs. 9 and 10. In regions 1 and 2 we have space-time points for which w > x (that is, ct > x), so that $c^2\tau^2 = w^2 - x^2 > 0$. The proper time is a real quantity, $c^2\tau^2$ being positive, in these regions. In regions 3 we have space-time points for which x > w (that is, x > ct), so that $c^2\tau^2 = w^2 - x^2 < 0$. The proper time is an imaginary quantity, $c^2\tau^2$ being negative, in these regions. However, the quantity σ is real here for $\sigma^2 = x^2 - w^2 > 0$ in regions 3. Hence, either τ or σ is real for any two events (i.e., the event at the origin and the event else-where in space-time) and either τ or σ may be called the space-time interval between the two events. When τ is real the interval is called "time-like"; when σ is real the interval is called "spacelike." Because σ and τ are invariant properties of two events, it does not depend at all on what inertial frame is used to specify the events whether the interval between them is spacelike or timelike.

In the spacelike region we can find a frame in which the two events are simultaneous, so that σ can be thought of as the spatial interval be-tween the events in that frame (i.e., $\sigma^2 = x^2 - w^2 = \sigma^2 = x'^2 - w'^2$. But w' = 0 in S' so that $\sigma = x'$). In the timelike region we can find a frame in which the two events occur at the same place, so that T can be thought of as the time interval between the events in that frame (i.e., $\tau^2 = t^2 - \frac{x^2}{c^2} = t'^2 - \frac{x'^2}{c^2}$). But x' = 0 in S' so that $\tau = t'$.

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What can we say about points on the 45° lines? For such points, x = w. Therefore, the proper time interval between two events on these lines vanishes, for $c^2 \tau^2 = w^2 - x^2 = 0$ if x = w. We have seen that such lines represent the world lines of light rays and give the limiting velocity (v = c) of relativity. On one side of these 45° lines (shaded regions in Fig. 9) the proper time interval is real, on the other side (unshaded regions), it is imaginary. An imaginary value of τ would correspond to a velocity in excess of c. But no signals can travel faster than c. All this is relevant to an interesting question that can be posed about the unshaded regions.

Causality

In this region, which we have called the Present, there is no absolute time order of events; event O may precede event Q in one frame but follow event Q in another frame. What does this do to our deep-seated notions of cause and effect? Does relativity theory negate the causality principle? To test cause and effect, we would have to examine the events at the same place so that we could say absolutely that Q followed O, or that O followed Q, in each instance. But in the Present, or spacelike, region these two events occur in such rapid succession that the time difference is less than the time needed by a light ray to traverse the spatial distance between two events. We cannot fix the time order of such events absolutely, for no signal can travel from one event to the other faster than c. In other words, no frame of reference exists with respect to which the two events occur at the same place; thus, we simply cannot test causality for such events even in principle. Therefore, there is no violation of the law of causality implied by the relative time order of O and events in the spacelike region. We can arrive at this same result by an argument other than this operational one. If the two events, O and Q, are related causally, then they must be capable of interacting physically. But no physical signal can travel faster than c so that events O and Q cannot interact physically. Hence, their time order is immaterial for they cannot be related causally. Events that can interact physically with O are in regions other than the Present. For such events, O and P, relativity gives an unambiguous time order. Therefore, relativity is completely consistent with the causality principle.

Reference:

Introduction to Special Relativity – Robert Resnick **

Concept of Modern Physics – Arthur Beiser

Internet