Leontief Input-Output Models

Background

- Professor Wassily Leontief, a Nobel Prize winner,* deals with this particular question: "What level of output should each of the *n* industries in an economy produce, in order that it will just be sufficient to satisfy the total demand for that product?"

Background

The rationale for the term ***input-output analysis:*** The output of any industry (say, the steel industry) is needed as an input in many other industries, or even for that industry itself; therefore the "correct" (i.e., shortage-free as well as surplus-free) level of steel output will depend on the input requirements of all the ***n*** industries.

- In turn, the output of many other industries will enter into the steel industry as inputs, and consequently the "correct' levels of the other products will in turn depend partly upon the input requirements of the steel industry.

Background

- In view of this interindustry dependence, any set of "correct" output levels for the *n* industries must be one that is consistent with all the input requirements in the economy, so that no bottlenecks will arise anywhere.
- In this light, it is clear that input-output analysis should be of great use in production planning, such as in planning for the economic development of a country or for a program of national defense.

Background

- Strictly speaking, input-output analysis is not a form of the general equilibrium analysis.
  - Although the interdependence of the various industries is emphasized, the "correct" output levels envisaged are those which satisfy technical input-output relationships rather than market equilibrium conditions.
- Nevertheless, the problem posed in input-output analysis also boils down to one of solving a system of simultaneous equations, and **matrix algebra** can again be of service.

Structure of an Input-Output Model

- Since an input-output model normally encompasses a large number of industries, its framework is quite complicated.
- To simplify the problem, the following assumptions are as a rule adopted:
  - (1) each industry produces only one homogeneous commodity
  - (2) each industry uses a fixed input ratio (or factor combination) for the production of its output; and
  - (3) production in every industry is subject to constant returns to scale, so that a k-fold change in every input will result in an exactly k-fold change in the output.

Structure of an Input-Output Model

- From these assumptions we see that, in order to produce each unit of the *j*th commodity, the input need for the *i*th commodity must be a fixed amount, which we shall denote by a_{ij}. Specifically, the production of each unit of the jth commodity will require a_{1j} (amount) of the first commodity, a_{2j} of the second commodity,…, and a_{nj} of the nth commodity.
- The first subscript refers to the input, and the second to the output: a_{ij} indicates how much of the *i*th commodity is used for the production of each unit of the *j*th commodity.)

Input-Output Coefficient Matrix

	Output				
Input	**I**	**II**	**III**	···	**N**
I	a_{11}	a_{12}	a_{13}	···	a_{1n}
II	a_{21}	a_{22}	a_{23}	···	a_{2n}
III	a_{31}	a_{32}	a_{33}	···	a_{3n}
⋮	⋮	⋮	⋮		⋮
N	a_{n1}	a_{n2}	a_{n3}	···	a_{nn}

Input-Output Coefficient Matrix

- For our purposes, we assume that prices are given
- Unit used: "a dollar's worth" of each commodity
- $\mathbf{a}_{32} = 0.35$ means that 35 cents' worth of the third commodity is required as an input for producing a dollar's worth of the second commodity.
- The $\mathbf{a}_{ij}$ symbol will be referred to as an ***input coefficient.***

- For an n-industry economy, the input coefficients can be arranged into a matrix $\boldsymbol{A} = [\mathbf{a}_{ij}]$, in which each ***column*** specifies the input requirements for the production of one unit of the output of a particular industry.
- The second column, for example, states that to produce a unit (a dollar's worth) of commodity II, the inputs needed are: $\mathbf{a}_{12}$ units of commodity I, $\boldsymbol{a}_{22}$ units of commodity II, etc. If no industry uses its own product as an input, then the elements in the principal diagonal of matrix $\boldsymbol{A}$ will all be zero.

The Open Model

- If the *n* industries in Table 5.2 constitute the entirety of the economy, then all their products would be for the sole purpose of meeting the ***input demand*** of the same *n* industries (to be used in further production) as against the ***final demand*** (such as consumer demand, not for further production).
- At the same time, all the inputs used in the economy would be in the nature of ***intermediate inputs*** (those supplied by the *n* industries) as against ***primary inputs*** (such as labor, not an industrial product). To allow for the presence of final demand and primary inputs, we must include in the model an ***open sector*** outside of the n-industry network. Such an open sector can accommodate the activities of the consumer households, the government sector, and even foreign countries.

The Open Model

- In view of the presence of the open sector, the sum of the elements in each column of the input-coefficient matrix ***A*** (or ***input matrix A*****,** for short) must be less than 1.
- Each column sum represents the *partial* input cost (not including the cost of primary inputs) incurred in producing a dollar's worth of some commodity;
- If this sum is greater than or equal to $1, therefore, production will not be economically justifiable.

The Open Model

- Symbolically, this fact may be stated thus:

$$\sum_{i=1}^{n} a_{ij} < 1 \qquad (j = 1, 2, \ldots, n)$$

- Where the summation where the summation is over *i*, that is, over the elements appearing in the various *rows* of a specific column *j*.

- Since the value of output ($1) must be fully absorbed by the payments to all factors of production, the amount by which the column sum falls short of $ 1 must represent the payment to the primary inputs of the open sector. Thus the value of the primary inputs needed to produce a unit of the *j*th commodity would be

$$1 - \sum_{i=1}^{n} a_{ij}$$

The Open Model

$$x_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}X_n + d_1$$

where d_1 denoes the final demand for its output

and $a_{1j}x_j$ is the input demand of the *j*th industry. Similarly,

$$x_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}X_n + d_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}X_n + d_1$$

The Open Model

After moving all terms that involve the variables x_j to the left of the equals signs, and leaving only the exogenously determined final demands d_j on the right, we can express the "correct" output levels of the *n* industries by the following system of *n* linear equations:

The Open Model

$$
\begin{aligned}
(1-a_{11})x_1 - \quad a_{12}x_2 - \cdots \quad - a_{1n}x_n &= d_1 \\
-a_{21}x_1 + (1-a_{22})x_2 - \cdots \quad - a_{2n}x_n &= d_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
-a_{n1}x_1 \quad - a_{n2}x_2 - \cdots + (1-a_{nn})x_n &= d_2
\end{aligned}
$$

In matrix notation,

$$
\begin{bmatrix}
(1-a_{11}) & -a_{12} & \cdots & -a_{1n} \\
-a_{21} & (1-a_{22}) & \cdots & -a_{2n} \\
\vdots & \vdots & & \vdots \\
-a_{n1} & -a_{n2} & \cdots & (1-a_{nn})
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}
$$

The Open Model

- If the 1s in the diagonal of the matrix on the left are ignored, the matrix is simply
  $-A=[-a_{ij}]$.
- The matrix is the sum of the identity matrix In and the matrix –A. Thus (5.20') can be written as

$$(I - A)x=d$$

The Open Model

$(I - A)x = d$

$x =$ variable vector

$d =$ final demand (constant term) vector

$I - A =$ Leontief matrix

If I - A is nonsingular, we can obtain its inverse and the unique solution is:

$$x^* = (I - A)^{-1} d$$

Numerical Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix}$$

Note that each column sum in A is less than 1. If a_{0j} is the dollar's worth of the jth commodity, we can write (subtract each column sum from 1):

$a_{01} = 0.3 \qquad a_{02} = 0.3 \qquad a_{03} = 0.4$

Numerical Example

The open input-output system can be expressed in the form

$(I - A)x = d$ as follows:

$$\begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ -0.1 & -0.3 & 0.8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Numerical Example

By inverting the 3x3 Leontief matrix, the solution would be:

$$\begin{bmatrix} x_1 * \\ x_2 * \\ x_3 * \end{bmatrix} = (I - A)^{-1} d = \frac{1}{0.384} \begin{bmatrix} 0.66 & 0.30 & 0.24 \\ 0.34 & 0.62 & 0.24 \\ 0.21 & 0.27 & 0.60 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

If the final-demand vector happens to be $\begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix}$, the solution values are:

$$x_1^* = \frac{1}{0.384}[0.66(10) + 0.30(5) + 24(6)] = \frac{9.54}{0.384} = 24.84$$

$$x_2^* = \frac{7.94}{0.384} = 20.68 \qquad x_3^* = \frac{7.05}{0.384} = 18.36$$