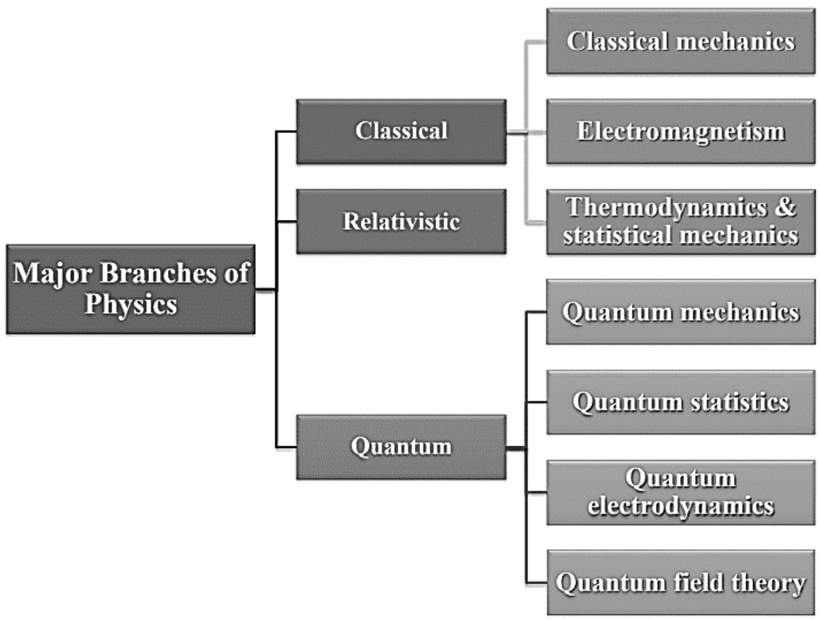
Preliminary lecture

Applied Electricity and Magnetism

PHY1221 75 marks [70%(52) Exam, 20% (15)Quizzes, 10% (7.5) Attendance]

Credit: 3, Contact hours/week: 3, Exam time: 3hrs,

Students should answer 6 questions out of 8 taking not more than three in one section.

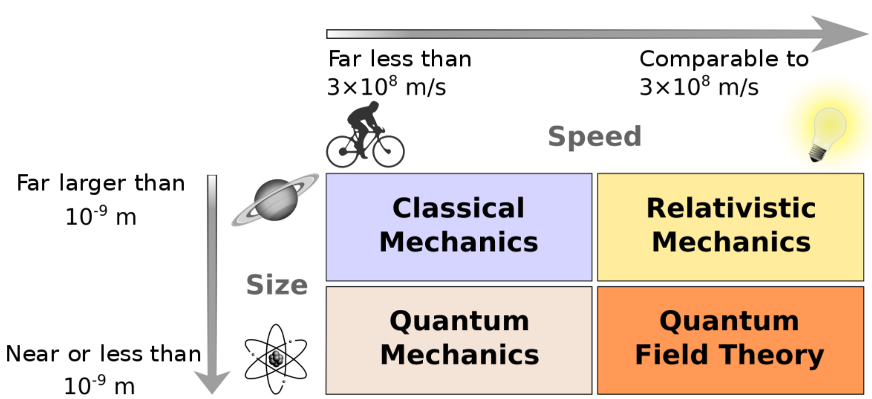


Consideration of size→ **Macroscopic** and **Microscopic**

Macroscopic: Star, planet, etc which visible to naked eye

Microscopic: Atoms, molecules, electron etc which visible with microscope

Road map in physics:



Section-A:

I. Electrostatics
II. Capacitors
III. Electric Current

Section-B:

I. Electromagnetic Induction
II. Thermoelectricity
III. DC and AC Circuits

- Classical physics → Two distinct aspect of nature
  - Matter: localized
  - Energy: Wave, spread in space

Classical Physics

Matter — Energy

Modern Physics

Classical physics
- Classical mechanics
- Electrodynamics

One body system

Modern Physics
- Quantum mechanics
- Relativity
- Statistical mechanics

Many body system

Reference:

Physics part 1 and part 11 – David Halliday and Robert Resnick

Concept of Modern Physics – Arthur Beiser

Electrostatics

Electric field

An **electric field** (sometimes abbreviated as **E-field**) is a vector field surrounding an electric charge that exerts force on other charges, attracting or repelling them. Mathematically the electric field is a vector field that associates to each point in space the force, called the Coulomb force, that would be experienced per unit of charge by an infinitesimal test charge at that point. The units of the electric field in the SI system are Newtons per coulomb (N/C), or volts per meter (V/m).
Electric fields are created by electric charges, or by time-varying magnetic fields. On an atomic scale, the electric field is responsible for the attractive force between the atomic nucleus and electrons that holds atoms together, and the forces between atoms that cause chemical bonding. Electric fields and magnetic fields are both manifestations of the electromagnetic force, one of the four fundamental forces (or interactions) of nature.

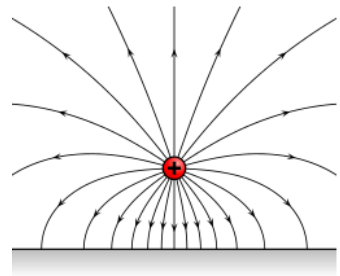


Fig.1: Electric field emanating from a point positive electric charge suspended over an infinite sheet of conducting material.

Electric Lines of Force:

An electric line of force is an imaginary continuous line or curve drawn in an electric field such that tangent to it at any point gives the direction of the electric force at that point. The direction of a line of force is the direction along which a small free positive charge will move along the line. Field lines directed into a closed surface are considered negative; those directed out of a closed surface are positive. If there is no net charge within a closed surface, every field line directed into the surface continues through the interior and is directed outward elsewhere on the surface.

Electric Flux:

Electric flux is a property of an electric field that may be thought of as the number of electric lines of force (or electric field lines) that intersect a given area. Electric field lines are considered to originate on positive electric charges and to terminate on negative charges. If a net charge is contained inside a closed surface, the total flux through the surface is proportional to the enclosed charge, positive if it is positive, negative if it is negative.

The mathematical relation between electric flux and enclosed charge is known as Gauss's law for the electric field, one of the fundamental laws of electromagnetism.

$$\Phi_E = \mathbf{E} \cdot \mathbf{S} = ES\cos\theta,$$

Where **E** is the electric field (having units of V/m), *E* is its magnitude, *S* is the area of the surface, and θ is the angle between the electric field lines and the normal (perpendicular) to *S*.

For a non-uniform electric field, the electric flux $d\Phi_E$ through a small surface area *d*S is given by

$$d\Phi_E = \mathbf{E} \cdot d\mathbf{S}$$

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Electric Dipole

An electric dipole is defined as a pair of equal and opposite charges separated by a distance. However, a continuous charge distribution can also be approximated as an electric dipole from a large distance. These dipoles are characterized by their dipole moment, a vector quantity defined as the charge multiplied by their separation and the direction of this vector quantity is from the -ve charge to the +ve charge. The total charge corresponding to a dipole is always zero. As the positive and negative charge centers are separated by a finite distance, the electric field at a test point does not cancel out completely leading to a finite electric field. Similarly, we also get finite electric potential due to a dipole.

Electric Dipole Potential:

The potential of an electric dipole can be found by superposing the point charge potentials of the two charges. Let us consider an electron dipole having charges q separated by distance d. Then the potential due to this dipole at point p at a distance r from the center of the dipole is:

$$V = kq\left[\frac{1}{r_+} - \frac{1}{r_-}\right] = kq\left[\frac{r_- - r_+}{r_+ r_-}\right]$$

Where $k = \frac{1}{4\pi\epsilon_0}$ is a constant.

For cases where r>>d, we can write $r_- - r_+ = dcos\theta$ and $r_- = r_+ = r$ this can be approximated by

$$V = kq\left[\frac{dcos\theta}{r^2}\right]$$

As we know, the dipole moment is $p = qd$, so

$$V = k\frac{pcos\theta}{r^2}$$

This is the required for the potential of an electric dipole.

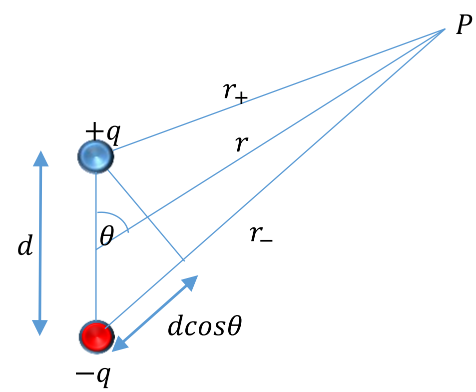


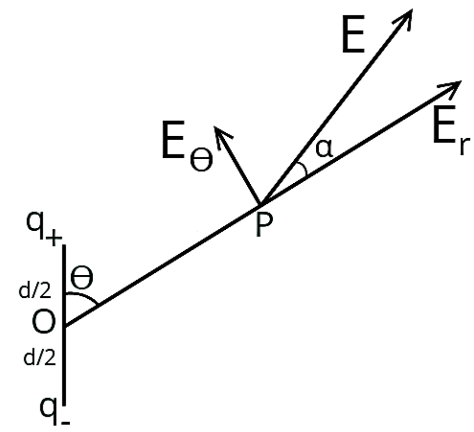
Fig. 1

Electric Field due to a Dipole

The electric field due to a pair of equal and opposite charges at any test point can be calculated using the Coulomb's law and the superposition principle. Let the test point P be at a distance r from the center of the dipole. The distance between +q and -q is d. We have shown the situation in the Fig. 1 above.

If $\mathbf{E}^+$ and $\mathbf{E}^-$ be the electric field at point P due to the positive and the negative charges separately then the total electric field **E** at Point P can be calculated by using the superposition principle.

Please note that the directions of $\boldsymbol{E}_+$ and $\boldsymbol{E}_-$ are along $\boldsymbol{r}_+$ and $\boldsymbol{r}_-$ respectively. This is the most general form of the electric field due to a dipole. However, we will express this vector in terms of radial and inclination vectors as shown in the diagram below.



In order to calculate the electric field in the polar coordinate, we will use the expression of the electric potential due to an electric dipole which we have calculated earlier.

$$V = k\frac{pcos\theta}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{pcos\theta}{r^2}$$

Here p=qd is the magnitude of the dipole moment. We can easily derive the electric field due to this dipole by calculating the negative gradient of this electric potential. In polar coordinate electric field will be independent of azimuthal (φ) coordinate.

$$\boldsymbol{E}_r = -\frac{\partial V}{\partial r} = \frac{1}{4\pi\epsilon_0}\frac{2p\ cos\theta}{r^3}$$

And

$$\boldsymbol{E}_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{1}{4\pi\epsilon_0}\frac{p\ sin\theta}{r^3}$$

Total electric field in vector form is

$$\boldsymbol{E} = \boldsymbol{E}_r + \boldsymbol{E}_\theta = \frac{p}{4\pi\epsilon_0}\left[\frac{2\ cos\theta}{r^3}\hat{r} + \frac{sin\theta}{r^3}\hat{\theta}\right]$$

And the magnitude of total electric field is

$$E = \sqrt{{E_r}^2 + {E_\theta}^2}$$

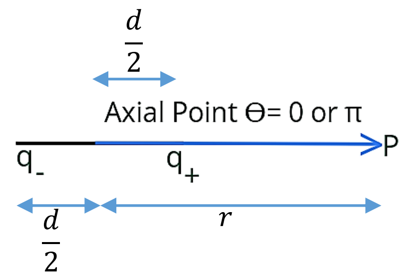
$$= \frac{p}{4\pi\epsilon_0 r^3}\sqrt{(2\ cos\theta)^2 + (sin\theta)^2}$$

$$= \frac{p}{4\pi\epsilon_0 r^3}\sqrt{3\ cos^2\theta + 1}$$

As shown in the diagram, the resultant electric field makes an angle α with the radial vector. Then the direction of the resultant is

$$\tan\alpha = \frac{E_\theta}{E_r} = \frac{\frac{1}{4\pi\epsilon_0}\frac{p\ sin\theta}{r^3}}{\frac{1}{4\pi\epsilon_0}\frac{2p\ cos\theta}{r^3}} = \frac{\tan\theta}{2}$$

Case:1: Electric field at an axial point;



In this case, the test point P is on the axis of the dipole. Consequently $\theta = 0 \; or \; \pi$. The electric field at point P is

$$E = \frac{p}{4\pi\epsilon_0 r^3}\sqrt{3\,cos^2 0 + 1}$$

$$= \pm\frac{2p}{4\pi\epsilon_0 r^3}$$

And the direction is

$$\tan\alpha = \frac{\tan 0}{2}$$

$$\alpha = tan^{-1}0 = 0°$$

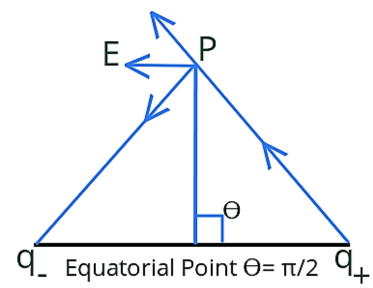
Case:2: Electric Field at an Equatorial Point

In this case, the test point P is on the perpendicular bisector of the dipole. $\theta = 90° \; or \; \pi/2$. The electric field at point P is

$$E = \frac{p}{4\pi\epsilon_0 r^3}\sqrt{3\,cos^2 0 + 1}$$

$$= \frac{p}{4\pi\epsilon_0 r^3}\sqrt{3\,cos^2 90 + 1}$$

$$= \frac{p}{4\pi\epsilon_0 r^3}$$



And the direction is

$$\tan\alpha = \frac{\tan 90}{2}$$

$$\alpha = tan^{-1}\left(\frac{\tan 90}{2}\right) = 180°$$

Electric Dipole in an Electric Field (Torque and potential energy)

<u>Torque</u>

We know the electric dipole moment of a dipole is defined as the vector p directed from -q to +q along the line joining the charges and having magnitude 2aq:

$$p \equiv 2aq \qquad (1)$$

Now suppose that an electric dipole is placed in a uniform electric field E, as shown in Figure 1. We denote E as the field external to the dipole, distinguishing it from the field due to the dipole. The field E is established by some other charge distribution, and we place the dipole into this field.

Let us imagine that the dipole moment makes an angle with the field. The electric forces acting on the two charges are equal in magnitude but opposite in direction as shown in Figure 1 (each has a magnitude $F = qE$. Thus, the net force on the dipole is zero. However, the two forces produce a net torque on the dipole; as a result, the dipole rotates in the direction that brings the dipole moment vector into greater alignment with the field. The torque due to the force on the positive charge about an axis through O in Figure 1 is Fa sinθ , where asinθ is the moment arm of F about O. This force tends to produce a clockwise rotation. The torque about O on the negative charge also is Fa sinθ; here again, the force tends to produce a clockwise rotation. Thus, the net torque about O is

$$\tau = 2Fasin\theta$$

Because $F = qE$ and $p = 2aq$ we can express τ as

$$\tau = 2aqEsin\theta$$

$$= pEsin\theta \qquad (2)$$

It is convenient to express the torque in vector form as the cross product of the vectors **p** and **E:**

$$\boldsymbol{\tau} = \boldsymbol{p} \times \boldsymbol{E}$$

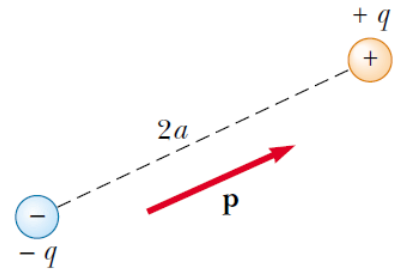


Fig. 1a: An electric dipole consists of two charges of equal magnitude but opposite sign separated by a distance of 2a. The electric dipole moment p is directed from - q to +q.

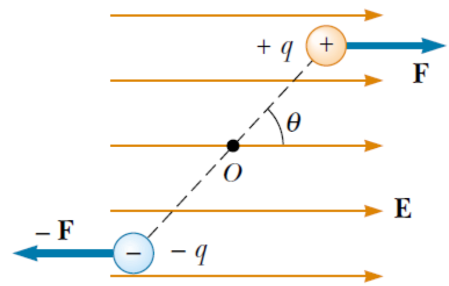


Fig. 1b: An electric dipole in a uniform external electric field. The dipole moment p is at an angle θ to the field, causing the dipole to experience a torque.

Potential Energy

We can determine the potential energy of the system of an electric dipole in an external electric field as a function of the orientation of the dipole with respect to the field. To do this, we recognize that work must be done by an external agent to rotate the dipole through an angle so as to cause the dipole moment vector to become less aligned with the field. The work done is then stored as potential energy in the system of the dipole and the external field. The work dW required to rotate the dipole through an angle dθ is $dW = \tau d\theta$. Because $\tau = pEsin\theta$ and because the work is transformed into potential energy U, we find that, for a rotation from θ_i to θ_f, the change in potential energy is

$$U_f - U_i = \int_{\theta_i}^{\theta_f} \tau\, d\theta = \int_{\theta_i}^{\theta_f} pE \sin\theta\, d\theta = pE\int_{\theta_i}^{\theta_f} \sin\theta\, d\theta$$

$$= pE\Big[-\cos\theta\Big]_{\theta_i}^{\theta_f} = pE(\cos\theta_i - \cos\theta_f)$$

The term that contains $\cos\theta_i$ is a constant that depends on the initial orientation of the dipole. It is convenient for us to choose $\theta_i = 0$, so that $\cos\theta_i = \cos 90 = 0$. Furthermore, let us choose $U_i = 0$ at $\theta_i = 90^o$ as our reference of potential energy. Hence, we can express a general value of $U = U_f$ as

$$U = -pE\cos\theta \tag{4}$$

We can write this expression for the potential energy of a dipole in an electric field as the dot product of the vectors **p** and **E:**

$$U = -\boldsymbol{p}.\boldsymbol{E} \tag{5}$$

Molecules are said to be polarized when a separation exists between the average position of the negative charges and the average position of the positive charges in the molecule. In some molecules, such as water, this condition is always present—such molecules are called polar molecules. Molecules that do not possess a permanent polarization are called nonpolar molecules.

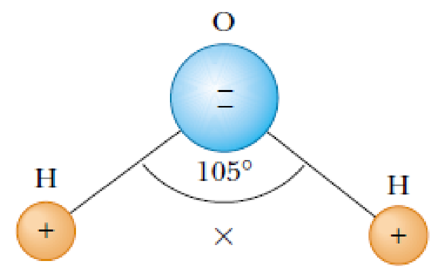


Fig. 2: The water molecule, H2O, has a permanent polarization resulting from its bent geometry. The center of the positive charge distribution is at the point x.

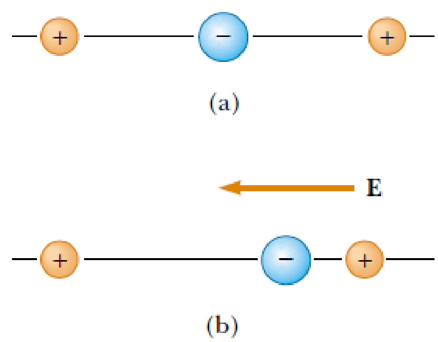


Fig. 3: (a) A symmetric molecule has no permanent polarization. (b) An external electric field induces a polarization in the molecule.

We can understand the permanent polarization of water by inspecting the geometry of the water molecule. In the water molecule, the oxygen atom is bonded to the hydrogen atoms such that an angle of 105° is formed between the two bonds (Fig. 2). The center of the negative charge distribution is near the oxygen atom, and the center of the positive charge distribution lies at a point midway along the line joining the hydrogen atoms (the point labeled in Fig. 2). We can model the water molecule and other polar molecules as dipoles because the average positions of the positive and negative charges act as point charges. As a result, we can apply our discussion of dipoles to the behavior of polar molecules.

Microwave ovens take advantage of the polar nature of the water molecule. When in operation, microwave ovens generate a rapidly changing electric field that causes the polar molecules to swing back and forth, absorbing energy from the field in the process. Because the jostling molecules collide with each other, the energy they absorb from the field is converted to internal energy, which corresponds to an increase in temperature of the food.

Another household scenario in which the dipole structure of water is exploited is washing with soap and water. Grease and oil are made up of nonpolar molecules, which are generally not attracted to water. Plain water is not very useful for removing this type of grime. Soap contains long molecules called surfactants. In a long molecule, the polarity characteristics of one end of the molecule can be different from those at the other end. In a surfactant molecule, one end acts like a nonpolar molecule and the other acts like a polar molecule. The nonpolar end can attach to a grease or *oil molecule, and the polar end can attach to a water molecule. Thus, the soap serves as a chain, linking the dirt and water molecules together. When the water is rinsed away, the grease and oil go with it.*

A symmetric molecule (Fig. 3a) has no permanent polarization, but polarization can be induced by placing the molecule in an electric field. A field directed to the left, as shown in Figure 3b, would cause the center of the positive charge distribution to shift to the left from its initial position and the center of the negative charge distribution to shift to the right. This induced polarization is the effect that predominates in most materials used as dielectrics in capacitors.

Stokes' theorem: $\oint_{\Gamma} \mathbf{F} \cdot d\mathbf{\Gamma} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$

[F is any function, $d\Gamma$ and dS are the small elements of length and surface respectively]

Divergence theorem: $\iiint_V (\nabla \cdot \mathbf{F})\, dV = \oiint_S (\mathbf{F} \cdot \mathbf{n})\, dS$

[**n** is the unit normal to the surface of volume V]

Gaussian Surface

A Gaussian surface is a closed imaginary surface in three-dimensional space through which the flux of a vector field is calculated. It enclosed all the charges for which flux is to be calculated.

Gauss's Law

The law was first formulated by Joseph-Louis Lagrange in 1773, followed by Carl Friedrich Gauss in 1813. **It is one of Maxwell's four equations**, which form the basis of classical electrodynamics.

The net electric flux through any hypothetical closed surface (Gaussian Surface) is equal to $\frac{1}{\varepsilon_0}$ times the net electric charge within that closed surface.

Gauss's law may be expressed as:

$$\Phi_E = \frac{Q}{\varepsilon_0} \qquad (1)$$

Where Φ_E is the electric flux through a closed surface *S* enclosing any volume *V*, *Q* is the total charge enclosed within *V*, and ε_0 is the electric permittivity.

Integral Form of Gauss's Law

The electric flux Φ_E is defined as a surface integral of the electric field:

$$\Phi_E = \oiint_S \mathbf{E} \cdot \mathbf{dA} \qquad (2)$$

Where **E** is the electric field, d**A** is a vector representing an infinitesimal element of area of the surface, and · represents the dot product of two vectors. Then we can write

$$Q = \varepsilon_0 \oiint_S \boldsymbol{E}.\boldsymbol{dA}$$

Since the flux is defined as an integral of the electric field, this expression of Gauss's law is called the integral form

Differential Form of Gauss's Law

By the divergence theorem, Gauss's law can alternatively be written in the *differential form*:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad (3)$$

Where $\nabla \cdot \mathbf{E}$ is the divergence of the electric field, ε_0 is the electric constant, and ρ is the total electric charge density (charge per unit volume).

Equivalence of integral and differential forms

The integral and differential forms are mathematically equivalent, by the divergence theorem.

Derivation of Gauss's Law

Let us consider a spherical surface of radius r containing an area element ΔA. The solid angle ΔΩ subtended at the center of the sphere by this element is defined to be

$$\Delta\Omega \equiv \frac{\Delta A}{r^2} \qquad (4)$$

From this equation, we see that ΔΩ has no dimensions because ΔA and r^2 both have dimensions L^2. The dimensionless unit of a solid angle is the steradian. Because the surface area of a sphere is $4\pi r^2$, the total solid angle subtended by the sphere is

$$\Omega = \frac{4\pi r^2}{r^2} = 4\pi \text{ steradians} \qquad (5)$$

Now consider a point charge q surrounded by a closed surface of arbitrary shape (Fig. 2). The total electric flux through this surface can be obtained by evaluating for each small area element ΔA and summing over all elements. The flux through each element is

$$\Delta\Phi_E = \mathbf{E} \cdot \Delta\mathbf{A} = E\,\Delta A \cos\theta = k_e q \frac{\Delta A \cos\theta}{r^2} \qquad (6)$$

Where r is the distance from the charge to the area element, θ is the angle between the electric field E and ΔA for the element, and $E = \frac{k_e q}{r^2}$ for a point charge. In Fig.3, we see that the projection of the area element perpendicular to the radius vector is $\Delta A cos\theta$. Thus, the quantity $\frac{\Delta A cos\theta}{r^2}$ is equal to the solid angle $\Delta\Omega$, that the surface element ΔA subtends at the charge q. We also see that $\Delta\Omega$ is equal to the solid angle subtended by the area element of a spherical surface of radius r. As the total solid angle at a point is 4π steradians, the total flux through the closed surface is

$$\Phi_E = k_e q \oint \frac{dA \cos\theta}{r^2} = k_e q \oint d\Omega = 4\pi k_e q = \frac{q}{\epsilon_0} \quad (7)$$

Thus we have derived Gauss's law. Note that this result is independent of the shape of the closed surface and independent of the position of the charge within the surface.

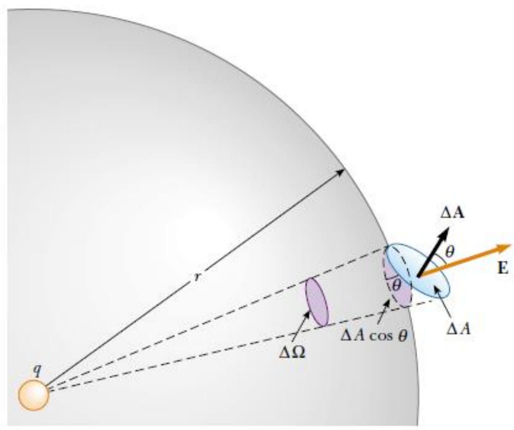


Fig. 3: The area element ΔA subtends a solid angle $\Delta\Omega = \frac{\Delta A cos\theta}{r^2}$ at the charge q.

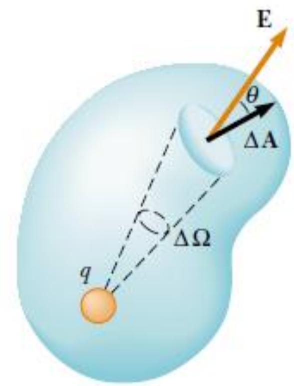


Fig2: A closed surface of arbitrary shape surrounds a point charge q. The net electric flux through the surface is independent of the shape of the surface.

Application of Gauss's Law to Charged Insulators

In choosing the surface, we should always take advantage of the symmetry of the charge distribution so that we can remove ***E*** from the integral and solve for it. The goal in this type of calculation is to determine a surface that satisfies one or more of the following conditions:

1. The value of the electric field can be argued by symmetry to be constant over the surface.

2. The dot product in ***E.dA*** can be expressed as a simple algebraic product *E dA* because **E** and *d***A** are parallel.

3. The dot product ***E.dA*** is zero when **E** and *d***A** are perpendicular.

4. The field can be argued to be zero over the surface.

The Electric Field Due to a Point Charge

Starting with Gauss's law, calculate the electric field due to an isolated point charge q.

Solution: A single charge represents the simplest possible charge distribution, and we use this familiar case to show how to solve for the electric field with Gauss's law. We choose a spherical Gaussian surface of radius *r* centered on the point charge, as shown in Fig.4 . The electric field due to a positive point charge is directed radially outward by symmetry and is therefore normal to the surface at every point. Thus, as in condition (2), **E** is parallel to ***d*A** at each point. Therefore, **E.dA**= E dA and Gauss's law gives

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \oint E\, dA = \frac{q}{\epsilon_0} \qquad (8)$$

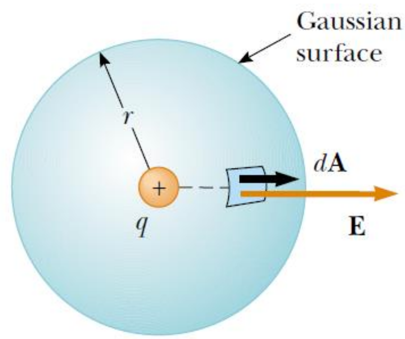


Fig. 4: The point charge q is at the center of the spherical Gaussian surface, and **E** is parallel to **dA** at every point on the surface

By symmetry, *E* is constant everywhere on the surface, which satisfies condition (1), so it can be removed from the integral. Therefore,

$$\oint E\, dA = E \oint dA = E(4\pi r^2) = \frac{q}{\epsilon_0} \qquad (9)$$

Where we have used the fact that the surface area of a sphere is $4\pi r^2$. Now, we solve for the electric field:

$$E = \frac{q}{4\pi\epsilon_0 r^2} = k_e \frac{q}{r^2} \qquad (10)$$

This is the familiar electric field due to a point charge that we found from the Coulomb's law.

A Spherically Symmetric Charge Distribution

An insulating solid sphere of radius a has a uniform volume charge density ρ and carries a total positive charge Q. (a) Calculate the magnitude of the electric field at a point outside the sphere.

(b) Find the magnitude of the electric field at a point inside the sphere.

Solution (a): As the charge distribution is spherically symmetric, we again select a spherical Gaussian surface of radius r, concentric with the sphere, as shown in Fig.5. For this choice, conditions (1) and (2) are satisfied, as they were for the point charge in previous example, we can write,

$$E = k_e \frac{Q}{r^2} \qquad (\text{for } r > a) \tag{11}$$

Note that this result is identical to the one we obtained for a point charge. Therefore, we conclude that, for a uniformly charged sphere, the field in the region external to the sphere is equivalent to that of a point charge located at the center of the sphere.

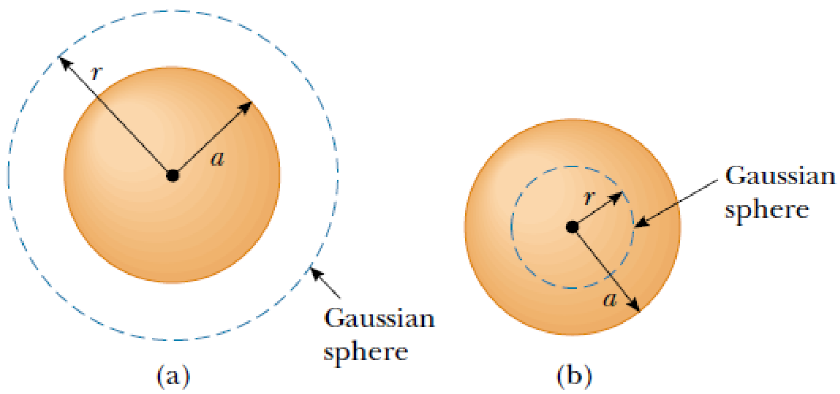


Fig.5: A uniformly charged insulating sphere of radius a and total charge Q.

Solution (b): In this case we select a spherical Gaussian surface having radius r < a, concentric with the insulated sphere (Fig. 5). Let us denote the volume of this smaller sphere by V'. To apply Gauss's law in this situation, it is important to recognize that the charge q_{in} within the Gaussian surface of volume V' is less than Q. To calculate q_{in}, we use the fact that $q_{in} = \rho V'$:

$$q_{in} = \rho V' = \rho \frac{4}{3}\pi r^3 \tag{12}$$

By symmetry, the magnitude of the electric field is constant everywhere on the spherical Gaussian surface and is normal to the surface at each point—both conditions (1) and (2) are satisfied. Therefore, Gauss's law in the region r < a gives

$$\oint E\, dA = E \oint dA = E(4\pi r^2) = \frac{q_{in}}{\epsilon_0} \tag{13}$$

Solving for E gives

$$E = \frac{q_{in}}{4\pi\epsilon_0 r^2} = \frac{\rho \frac{4}{3}\pi r^3}{4\pi\epsilon_0 r^2} = \frac{\rho}{3\epsilon_0} r \tag{14}$$

As $\rho = \frac{Q}{\frac{4}{3}\pi r^3}$ by definition and since $k_e = \frac{1}{4\pi r^2}$ this expression for E can be written as

$$E = \frac{Qr}{4\pi\epsilon_0 a^3} = \frac{k_e Q}{a^3} r \qquad (\text{for } r < a) \tag{15}$$

Note that this result for E differs from the one we obtained in part (a). It shows that E ⟶ 0 as r ⟶ 0. Therefore, the result eliminates the problem that would exist at r = 0 if E varied as $1/r^2$ inside the sphere as it does outside the sphere. That is, if E∝ 1/r2 for r < a, the field would be infinite at r = 0, which is physically impossible. Note also that the expressions for parts (a) and (b) match when r = a. A plot of E versus r is shown in Fig.6.

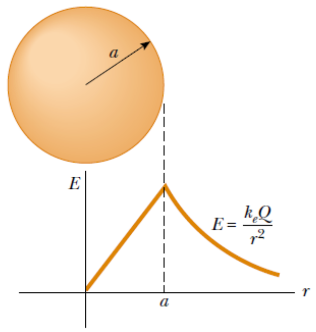


Fig. 6: A plot of E versus r for a uniformly charged insulating sphere. The electric field inside the sphere (r<a) varies linearly with r. The field outside the sphere (r>a) is the same as that of a point charge Q located at r = 0.

The Electric Field Due to a Thin Spherical Shell

A thin spherical shell of radius a has a total charge Q distributed uniformly over its surface (Fig.7). Find the electric field at points (a) outside and (b) inside the shell.

Solution (a): The calculation for the field outside the shell is identical to that for the solid sphere shown in Example 2a. If we construct a spherical Gaussian surface of radius r > a concentric with the shell (Fig. 7b), the charge inside this surface is Q. Therefore, the field at a point outside the shell is equivalent to that due to a point charge Q located at the center:

$$E = k_e \frac{Q}{r^2} \qquad \text{(for } r > a\text{)} \tag{16}$$

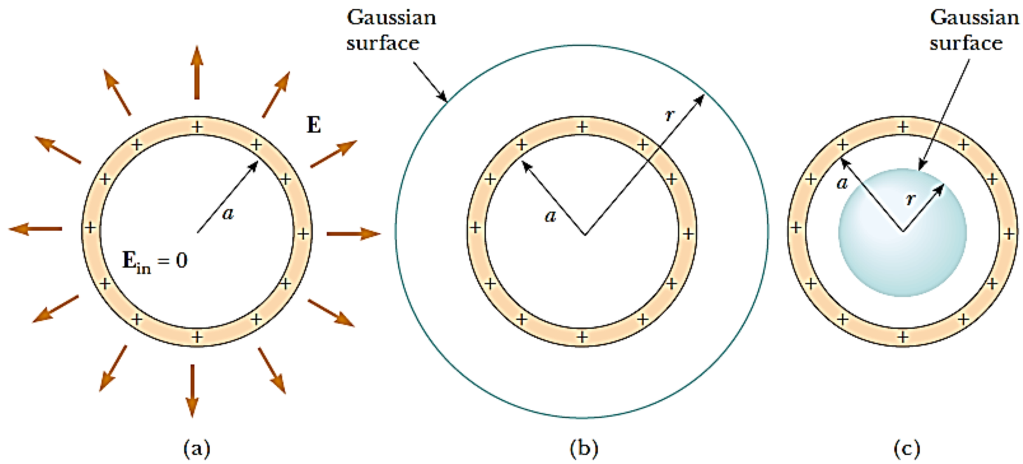


Fig. 7: A thin spherical shell of radius a and charge Q.

Solution (b): The electric field inside the spherical shell is zero. This follows from Gauss's law applied to a spherical surface of radius r < a concentric with the shell (Fig. 7c). Because of the spherical symmetry of the charge distribution and because the net charge inside the surface is zero—satisfaction of conditions (1) and (2) again—application of Gauss's law shows that E = 0 in the region r < a. We obtain the same results using coulombs law. This calculation is rather complicated. Gauss's law allows us to determine these results in a much simpler way.

A Cylindrically Symmetric Charge Distribution

Find the electric field a distance r from a line of positive charge of infinite length and constant charge per unit length λ

Solution: The symmetry of the charge distribution requires that **E** be perpendicular to the line charge and directed outward, as shown in Fig.8a and b. To reflect the symmetry of the charge distribution, we select

a cylindrical Gaussian surface of radius r and length l that is coaxial with the line charge. For the curved part of this surface, **E** is constant in magnitude and perpendicular to the surface at each point—satisfaction of conditions (1) and (2). Furthermore, the flux through the ends of the Gaussian cylinder is zero because **E** is parallel to these surfaces—the first application we have seen of condition (3).
We take the surface integral in Gauss's law over the entire Gaussian surface. Because of the zero value of **E.**d**A** for the ends of the cylinder, however, we can restrict our attention to only the curved surface of the cylinder. The total charge inside our Gaussian surface is λl. Applying Gauss's law and conditions (1) and (2), we find that for the curved surface

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = E \oint dA = EA = \frac{q_{\text{in}}}{\epsilon_0} = \frac{\lambda \ell}{\epsilon_0} \quad (17)$$

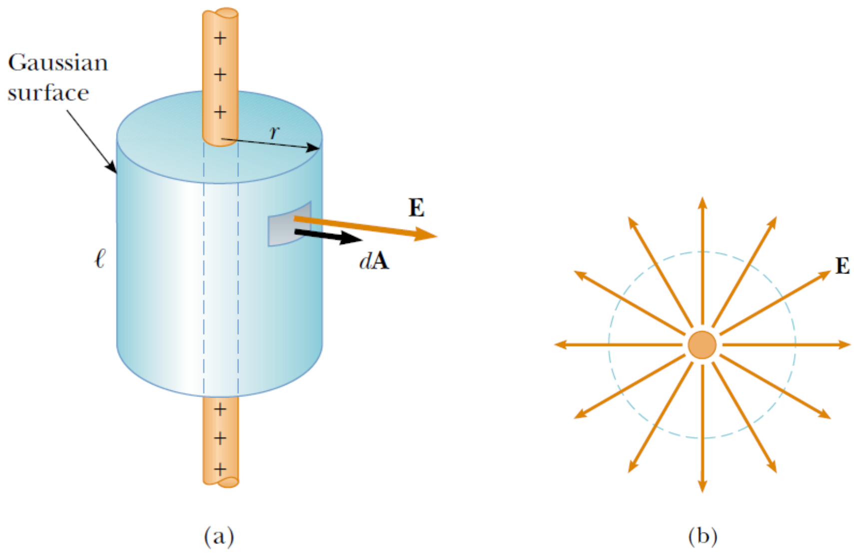


Fig. 8: (a) An infinite line of charge surrounded by a cylindrical Gaussian surface concentric with the line. (b) An end view shows that the electric field at the cylindrical surface is constant in magnitude and perpendicular to the surface.

The area of the curved surface is A=$2\pi r l$ therefore,

$$E(2\pi r \ell) = \frac{\lambda \ell}{\epsilon_0} \quad (18)$$

$$E = \frac{\lambda}{2\pi\epsilon_0 r} = 2k_e \frac{\lambda}{r} \quad (19)$$

Thus, we see that the electric field due to a cylindrically symmetric charge distribution varies as 1/r, whereas the field external to a spherically symmetric charge distribution varies as $1/r^2$. If the line charge in this example were of finite length, the result for E would not be that given by the above equation. A finite line charge does not possess sufficient symmetry for us to make use of Gauss's law. This is because the magnitude of the electric field is no longer constant over the surface of the Gaussian cylinder—the field near the ends of the line would be different from that far from the ends. Thus, condition (1) would not be satisfied in this situation. Furthermore, E is not perpendicular to the cylindrical surface at all points—the field vectors near the ends would have a component parallel to the line. Thus, condition (2) would not be satisfied.

A Nonconducting Plane of Charge

Find the electric field due to a nonconducting, infinite plane of positive charge with uniform surface charge density σ.

Solution:

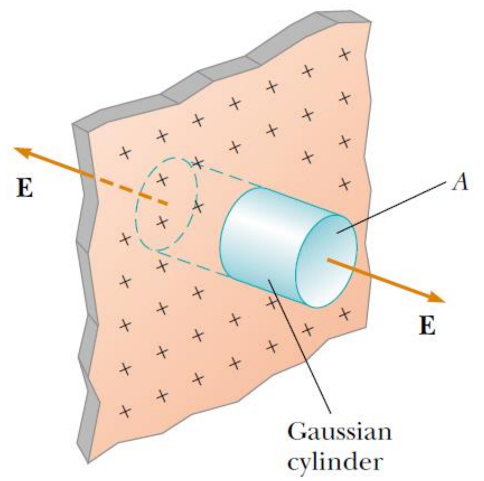


Fig. 9: A cylindrical gaussian surface penetrating an infinite plane of charge

By symmetry, **E** must be perpendicular to the plane and must have the same magnitude at all points equidistant from the plane. The fact that the direction of **E** is away from positive charges indicates that the direction of **E** on one side of the plane must be opposite its direction on the other side, as shown in Fig. 9. A Gaussian surface that reflects the symmetry is a small cylinder whose axis is perpendicular to the plane and whose ends each have an area A and are equidistant from the plane. Because E is parallel to the curved surface—and, therefore, perpendicular to d**A** everywhere on the surface—condition (3) is satisfied and there is no contribution to the surface integral from this surface. For the flat ends of the cylinder, conditions (1) and (2) are satisfied. The flux through each end of the cylinder is EA; hence, the total flux through the entire Gaussian surface is just that through the ends, $\Phi_E = 2EA$.

Noting that the total charge inside the surface is $q_{in} = \sigma A$, we use Gauss's law and find that

$$\Phi_E = 2EA = \frac{q_{\text{in}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0} \qquad (20)$$

$$E = \frac{\sigma}{2\epsilon_0} \qquad (21)$$

Because the distance from each flat end of the cylinder to the plane does not appear in the above equation, we conclude that E = $\frac{\sigma}{2\varepsilon_0}$ at any distance from the plane. That is, the field is uniform everywhere. An important charge configuration related to this example consists of two parallel planes, one positively charged and the other negatively charged, and each with a surface charge density σ. In this situation, the electric fields due to the two planes add in the region between the planes, resulting in a field of magnitude $\frac{\sigma}{\varepsilon_0}$, and cancel elsewhere to give a field of zero.

A Sphere Inside a Spherical Shell

A solid conducting sphere of radius a carries a net positive charge 2Q. A conducting spherical shell of inner radius b and outer radius c is concentric with the solid sphere and carries a net charge -Q. Using Gauss's law, find the electric field in the regions labeled 1,2,3 and 4 in Fig.10 and the charge distribution on the shell when the entire system is in electrostatic equilibrium.

Solution:

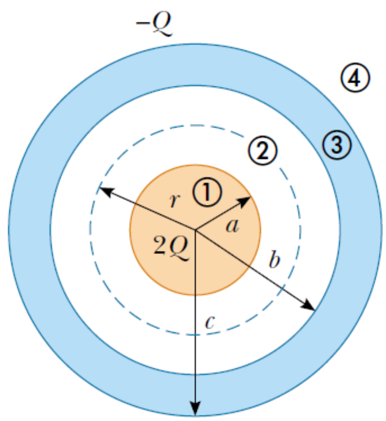


Fig. 10: A solid conducting sphere of radius a and carrying a charge 2Q surrounded by a conducting spherical shell carrying a charge -Q.

First note that the charge distributions on both the sphere and the shell are characterized by spherical symmetry around their common center. To determine the electric field at various distances r from this center, we construct a spherical Gaussian surface for each of the four regions of interest. Such a surface for region 2 is shown in Fig. 10.

To find E inside the solid sphere (region 1), consider a Gaussian surface of radius r < a. Because there can be no charge inside a conductor in electrostatic equilibrium, we see that $q_{in} = 0$; thus, on the basis of Gauss's law and symmetry, $E_1=0$ for r < a.

In region 2—between the surface of the solid sphere and the inner surface of the shell—we construct a spherical Gaussian surface of radius r where a < r < b and note that the charge inside this surface is +2Q (the charge on the solid sphere). Because of the spherical symmetry, the electric field lines must be directed radially outward and be constant in magnitude on the Gaussian surface. Following Example and using Gauss's law, we find that

$$E_2 A = E_2(4\pi r^2) = \frac{q_{\text{in}}}{\epsilon_0} = \frac{2Q}{\epsilon_0} \quad (22)$$

$$E_2 = \frac{2Q}{4\pi\epsilon_0 r^2} = \frac{2k_e Q}{r^2} \qquad (\text{for } a < r < b) \quad (23)$$

In region 4, where r > c, the spherical Gaussian surface we construct surrounds a total charge of q_{in} = +2Q-Q = + Q. Therefore, application of Gauss's law to this surface gives

$$E_4 = \frac{k_e Q}{r^2} \qquad (\text{for } r > c)$$

In region 3, the electric field must be zero because the spherical shell is also a conductor in equilibrium. If we construct a Gaussian surface of radius r where b < r < c, we see that q_{in} must be zero because From this argument, we conclude that the charge on the inner surface of the spherical shell must be +2Q to cancel the charge -2Q on the solid sphere. Because the net charge on the shell is -Q, we conclude that its outer surface must carry a charge +Q.

ELECTRIC POTENTIAL DUE TO CONTINUOUS CHARGE DISTRIBUTIONS

We can calculate the electric potential due to a continuous charge distribution in two ways. If the charge distribution is known, we can calculate the potential for every charges and then sum over the potentials to get the total potential due to the distribution.

Or we can consider the potential due to a small charge element dq, treating this element as a point charge (Fig. 11). The electric potential dV at some point P due to the charge element dq is

$$dV = k_e \frac{dq}{r} \quad (24)$$

Where r is the distance from the charge element to point P. To obtain the total potential at point P, we integrate this equation to include contributions from all elements of the charge distribution. Because each element is, in general, a different distance from point P and because k_e is constant, we can express V as

$$V = k_e \int \frac{dq}{r} \quad (25)$$

Note that, this expression for V uses a particular reference: The electric potential is taken to be zero when point P is infinitely far from the charge distribution. If the charge distribution is highly symmetric, we first evaluate **E** at any point using Gauss's law and then substitute the value obtained into equation

(26)

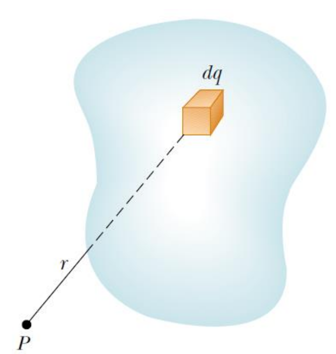


Fig. 11: The electric potential at the point P due to a continuous charge distribution

$$\Delta V = \frac{\Delta U}{q_0} = -\int_A^B \mathbf{E} \cdot d\mathbf{s}$$

to determine the potential difference ΔV between any two points. We then choose the electric potential V to be zero at some convenient point.

Electric Potential Due to a Uniformly Charged Ring

(a)Find an expression for the electric potential at a point P located on the perpendicular central axis of a uniformly charged ring of radius a and total charge Q.

(b) Find an expression for the magnitude of the electric field at point P.

Solution (a): Let us orient the ring so that its plane is perpendicular to an x axis and its center is at the origin. We can then take point P to be at a distance x from the center of the ring, as shown in Fig.12. The charge element dq is at a distance $\sqrt{x^2 + a^2}$ from point P. Hence, we can express V as

$$V = k_e \int \frac{dq}{r} = k_e \int \frac{dq}{\sqrt{x^2 + a^2}} \quad (27)$$

Because each element dq is at the same distance from point P we can remove $\sqrt{x^2 + a^2}$ from the integral, and V reduces to

$$V = \frac{k_e}{\sqrt{x^2 + a^2}} \int dq = \frac{k_e Q}{\sqrt{x^2 + a^2}} \quad (28)$$

The only variable in this expression for V is x. This is not surprising because our calculation is valid only for points along the x axis, where y and z are both zero.

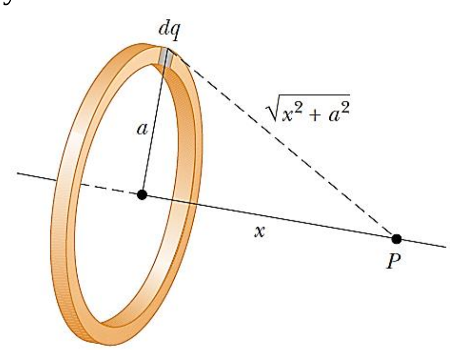


Fig. 12: A uniformly charged ring of radius a lies in a plane perpendicular to the x axis. All segments dq of the ring are the same distance from any point P lying on the x axis.

Solution (b): From symmetry, we see that, along the x axis **E** can have only an x component. Therefore, we can write

$$E_x = -\frac{dV}{dx} = -k_eQ\frac{d}{dx}(x^2 + a^2)^{-1/2}$$

$$= -k_eQ(-\tfrac{1}{2})(x^2 + a^2)^{-3/2}(2x)$$

$$= \frac{k_eQx}{(x^2 + a^2)^{3/2}} \quad (29)$$

This result agrees with that obtained by direct integration. Note that $E_x = 0$ at x = 0 (the center of the ring). Could you have guessed this from Coulomb's law?

Exercise: What is the electric potential at the center of the ring? What does the value of the field at the center tell you about the value of V at the center?
Answer: $V = k_eQ/a$. Because Ex = - dV/dx = 0 at the center, V has either a maximum or minimum value; it is, in fact, a maximum.

Electric Potential Due to a Uniformly Charged Disk

Find (a) the electric potential and (b) the magnitude of the electric field along the perpendicular central axis of a uniformly charged disk of radius a and surface charge density σ.

Solution (a): Again, we choose the point P to be at a distance x from the center of the disk and take the plane of the disk to be perpendicular to the x axis. We can simplify the problem by dividing the disk into a series of charged rings. The electric potential of each ring is given by Equation 28. Consider one such ring of radius r and width dr, as indicated in Fig.13. The surface area of the ring is $dA = 2\pi r dr$;

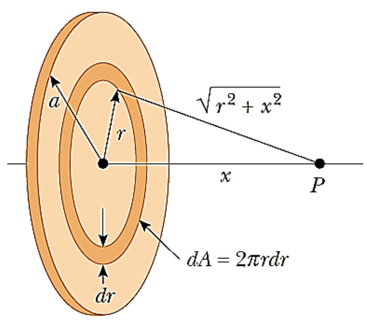


Fig. 13: A uniformly charged disk of radius a lies in a plane perpendicular to the x axis.

From the definition of surface charge density, we know that the charge on the ring is $dq = \sigma dA = \sigma 2\pi r dr$. Hence, the potential at the point P due to this ring is

$$dV = \frac{k_e\, dq}{\sqrt{r^2 + x^2}} = \frac{k_e\, \sigma 2\pi r\, dr}{\sqrt{r^2 + x^2}}$$

To find the total electric potential at P, we sum over all rings making up the disk. That is, we integrate dV from r = 0 to r = a:

$$V = \pi k_e \sigma \int_0^a \frac{2r\, dr}{\sqrt{r^2 + x^2}} = \pi k_e \sigma \int_0^a (r^2 + x^2)^{-1/2}\, 2r\, dr$$

This integral is of the form $u^n du$ and has the value $un^{+}1/(n+1)$, where n= -1/2 and $u=r^2+x^2$. This gives

$$V = 2\pi k_e \sigma[(x^2 + a^2)^{1/2} - x] \quad (30)$$

Solution (b): As in Example 1, we can find the electric field at any axial point from

$$E_x = -\frac{dV}{dx} = 2\pi k_e \sigma\left(1 - \frac{x}{\sqrt{x^2 + a^2}}\right) \quad (31)$$

The calculation of V and **E** for an arbitrary point off the axis is more difficult to perform, and we do not treat this situation in this text.

Electric Potential Due to a Finite Line of Charge

A rod of length l located along the x axis has a total charge Q and a uniform linear charge density $\lambda = Q/l$. Find the electric potential at a point P located on the y axis a distance a from the origin.

Solution: The length element dx has a charge dq = λ dx. Because this element is a distance r=$\sqrt{x^2 + a^2}$ from point P, we can express the potential at point P due to this element as

$$dV = k_e \frac{dq}{r} = k_e \frac{\lambda\, dx}{\sqrt{x^2 + a^2}}$$

To obtain the total potential at P, we integrate this expression over the limits x = 0 to x = l. Noting that k_e and λ are constants, we find that

$$V = k_e \lambda \int_0^\ell \frac{dx}{\sqrt{x^2 + a^2}} = k_e \frac{Q}{\ell} \int_0^\ell \frac{dx}{\sqrt{x^2 + a^2}}$$

This integral has the following value

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2})$$

Evaluating V, we find that

$$V = \frac{k_e Q}{\ell} \ln\left(\frac{\ell + \sqrt{\ell^2 + a^2}}{a}\right) \quad (32)$$

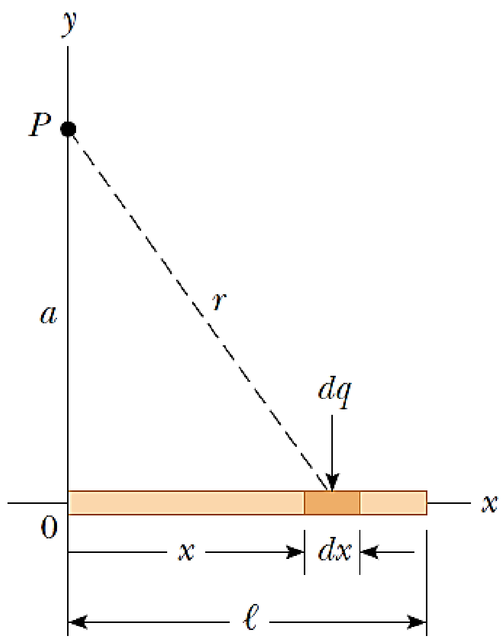


Fig. 14: A uniform line charge of length located along the x axis.