Rigid Bodies

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Rigid Body:

A rigid body is defined as a system of mass points subjected to the holonomic constraints (constraint relations are or can be made independent of velocities) that the distance between all pairs of pints remain constant through the motion, i.e. a rigid body means a rigid assembly of particles with fixed inter-particle distances.

By a rigid body we mean a rigid assembly of particles, with fixed inter-particle distances. Thus we always neglected the deformations that occur in actual bodies. In order to discuss the mechanics of a rigid body, we must have means of describing its position and orientation. A rigid body with N particles can almost have 3N degree of freedom, three for each constituent particle but because of the large number of constraints, the number of degrees of freedom or independent co-ordinates required to describe its motion will be much less than 3N

Number of Necessary independent coordinates for rigid body:

A rigid body with N particles can at most have 3N degrees of freedom, but these are greatly reduced by the constraints, which can be expressed as equations of the form

$$r_{ij} = c_{ij} \qquad (1)$$

where r_{ij} is the distance between the ith and jth particles and the c's are constants. The actual number of degrees of freedom cannot be obtained simply by subtracting the number of constraint equations from 3N, for there are $\frac{1}{2}N(N-1)$ possible equations of the form of Eq. (1), which is far greater than 3N for large N.

Therefore, [reduced number of degrees of freedom] = [Number of degrees of freedom of a rigid body almost have] – [Number of constraint equation]

$$= 3N - \frac{1}{2}N(N-1) \qquad (2)$$

In truth, the Eqs. (1) are not all independent. To fix a point in the rigid body, it is not necessary to .specify its distances to all other points in the body; we need only state the distances to any three other non-collinear points (Fig.1).

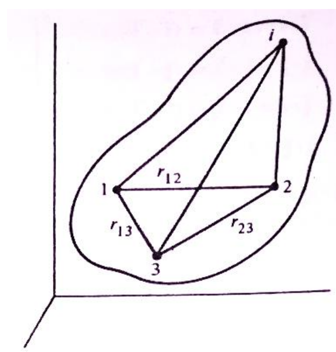

Fig.1: The location of a point in a rigid body by its distances from three reference points.

Thus, once the positions of three of the particles of the rigid body are determined, the constraints fix the positions of all remaining particles. The number of degrees of freedom therefore cannot be more than nine.

But the three reference points are themselves not independent: there are in fact three equations of rigid constraint imposed on them.

$$r_{12} = c_{12} \qquad r_{23} = c_{23} \qquad r_{13} = c_{13}$$

that reduce the number of degrees of freedom to six. **That only six coordinates are needed can also be seen from the following considerations.**

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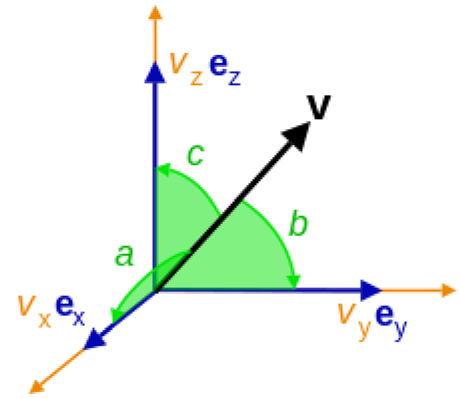
Direction Cosines: The **direction cosines** (or **directional cosines**) of a vector are the cosines of the angles between the vector and the three coordinate axes. Equivalently, they are the contributions of each component of the basis to a unit vector in that direction. Direction cosines are an analogous extension of the usual notion of slope to higher dimensions.



More generally, **direction cosine** refers to the cosine of the angle between any two vectors. They are useful for forming direction cosine matrices that express one set of orthonormal basis vectors in terms of another set, or for expressing a known vector in a different basis.

If **v** is a Euclidean vector in three-dimensional Euclidean space

$$\mathbf{v} = v_x \mathbf{e}_x + v_y \mathbf{e}_y + v_z \mathbf{e}_z$$

where $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ are the standard basis in Cartesian notation, then the direction cosines are

$$\alpha = \cos a = \frac{\mathbf{v}\cdot\mathbf{e}_x}{\|\mathbf{v}\|} = \frac{v_x}{\sqrt{v_x^2+v_y^2+v_z^2}}$$

$$\beta = \cos b = \frac{\mathbf{v}\cdot\mathbf{e}_y}{\|\mathbf{v}\|} = \frac{v_y}{\sqrt{v_x^2+v_y^2+v_z^2}}$$

$$\gamma = \cos c = \frac{\mathbf{v}\cdot\mathbf{e}_z}{\|\mathbf{v}\|} = \frac{v_z}{\sqrt{v_x^2+v_y^2+v_z^2}}$$

It follows that by squaring each equation and adding the results

$$\cos^2 a + \cos^2 b + \cos^2 c = \alpha^2 + \beta^2 + \gamma^2 = 1.$$

Here α, β and γ are the direction cosines and the Cartesian coordinates of the unit vector **v**/|**v**|, and a, b and c are the direction angles of the vector **v**.

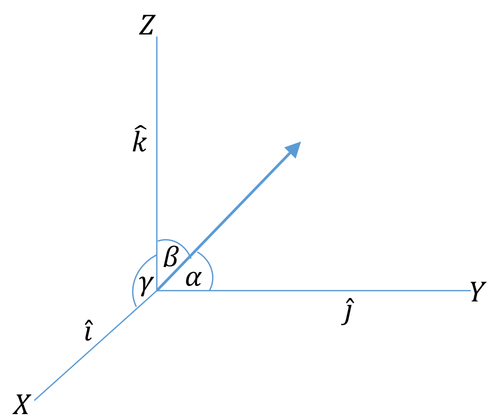
Prove that for any vector $\vec{A}$

a) $\vec{A} = (\vec{A}.\hat{\imath})\hat{\imath} + (\vec{A}.\hat{\jmath})\hat{\jmath} + (\vec{A}.\hat{k})\hat{k}$

b) $\vec{A} = A(cos\alpha\ \hat{\imath} + cos\beta\ \hat{\jmath} + cos\gamma\ \hat{k})$

Let a vector **A**, which can be expressed as,

$$\boldsymbol{A} = \vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$$



We know,

$\boldsymbol{A}.\hat{\imath} = |\boldsymbol{A}||\hat{\imath}|cos\alpha = Acos\alpha = A_1$

$\boldsymbol{A}.\hat{\jmath} = |\boldsymbol{A}||\hat{\jmath}|cos\beta = Acos\beta = A_2$

$\boldsymbol{A}.\hat{k} = |\boldsymbol{A}||\hat{k}|cos\gamma = Acos\gamma = A_3$

Putting the values of components of vector **A** we get,

$$\vec{A} = (\vec{A}.\hat{\imath})\hat{\imath} + (\vec{A}.\hat{\jmath})\hat{\jmath} + (\vec{A}.\hat{k})\hat{k}$$

And

$$\vec{A} = Acos\alpha\,\hat{\imath} + Acos\beta\,\hat{\jmath} + Acos\gamma\,\hat{k}$$

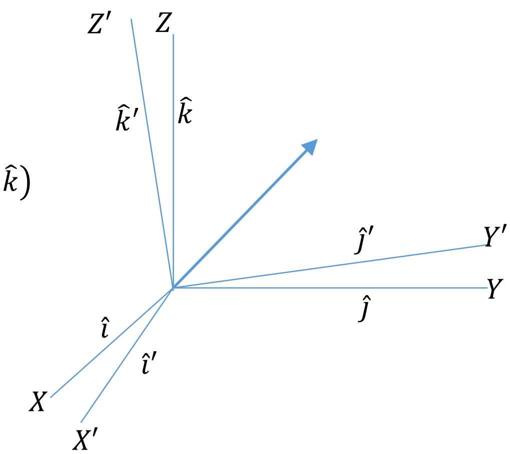
$$= A(cos\alpha\,\hat{\imath} + cos\beta\,\hat{\jmath} + cos\gamma\,\hat{k})$$

Orthogonality Condition for Linear Transformation Matrix:

Let α_i, β_i, γ_i be the direction cosines and i, j, k and i', j', k' be the unit vectors along the axes x y z fixed in space and x' y' z' axes fixed in the body. Then the position vector in these frames

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\vec{r} = x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}'$$



as the vector must be same in both frames.

Now we know, $\vec{A} = |\boldsymbol{A}|(cos\alpha\,\hat{\imath} + cos\beta\,\hat{\jmath} + cos\gamma\,\hat{k}) = |\boldsymbol{A}|(\alpha_1\hat{\imath} + \alpha_2\hat{\jmath} + \alpha_3\hat{k})$

Similarly we can write

$$\hat{\imath}' = |\hat{\imath}'|(\alpha_1\hat{\imath} + \alpha_2\hat{\jmath} + \alpha_3\hat{k})$$

$$= \alpha_1\hat{\imath} + \alpha_2\hat{\jmath} + \alpha_3\hat{k}$$

$$\hat{\jmath}' = \beta_1\hat{\imath} + \beta_2\hat{\jmath} + \beta_3\hat{k}$$

$$\hat{k}' = \gamma_1\hat{\imath} + \gamma_2\hat{\jmath} + \gamma_3\hat{k}$$

Now,

$$x' = \vec{r}.\hat{\imath}'$$

$$= (x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}').\hat{\imath}'$$

$$= (x\hat{\imath} + y\hat{\jmath} + z\hat{k}).\hat{\imath}'$$

$$= x(\hat{\imath}.\hat{\imath}') + y(\hat{\jmath}.\hat{\imath}') + z(\hat{k}.\hat{\imath}')$$

$$\Rightarrow x' = \alpha_1 x + \alpha_2 y + \alpha_3 z$$

Similarly,

$$y' = \beta_1 x + \beta_2 y + \beta_3 z$$

$$z' = \gamma_1 x + \gamma_2 y + \gamma_3 z \quad (1)$$

These equations are called transformation equation from space coordinate to body coordinate.

Reverse transformation equations are,

$$\left.\begin{aligned} x &= \alpha_1 x' + \beta_1 y' + \gamma_1 z' \\ y &= \alpha_2 x' + \beta_2 y' + \beta_2 z' \\ z &= \alpha_3 x' + \beta_3 y' + \gamma_3 z' \end{aligned}\right\} \qquad (2)$$

From eqn. 1 we get

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ y \end{bmatrix} \qquad (3)$$

Similarly from eqn. 2

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \qquad (4)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = I \begin{bmatrix} x \\ y \\ y \end{bmatrix}$$

As $I = AA^{-1}$ for orthogonal property.

$$\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} = I$$

$$\begin{bmatrix} {\alpha_1}^2 + {\beta_1}^2 + {\gamma_1}^2 & \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 & \alpha_1\alpha_3 + \beta_1\beta_3 + \gamma_1\gamma_3 \\ \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 & {\alpha_2}^2 + {\beta_2}^2 + {\gamma_2}^2 & \alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3 \\ \alpha_1\alpha_3 + \beta_1\beta_3 + \gamma_1\gamma_3 & \alpha_1\alpha_3 + \beta_1\beta_3 + \gamma_1\gamma_3 & {\alpha_3}^2 + {\beta_3}^2 + {\gamma_3}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So,

$${\alpha_1}^2 + {\beta_1}^2 + {\gamma_1}^2 = {\alpha_2}^2 + {\beta_2}^2 + {\gamma_2}^2 = {\alpha_3}^2 + {\beta_3}^2 + {\gamma_3}^2 = 1$$

$$\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = \alpha_1\alpha_3 + \beta_1\beta_3 + \gamma_1\gamma_3 = \alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3 = 0$$

We can write in general, as

$$\begin{aligned} \alpha_l\alpha_m + \beta_l\beta_m + \gamma_l\gamma_m &= 0\,; \qquad for\ l \neq m \\ &= 1\,; \qquad for\ l = m \\ &= \delta_{lm} \qquad (\,Kronecker\ Delta) \end{aligned}$$

Now, let us change the rotation and denote all coordinates by x, then we get,

$$x' \longrightarrow x_1'$$
$$y' \longrightarrow x_2'$$
$$z' \longrightarrow x_3'$$

Equation 1 and 2 constitute a group of transformation equation from a set of coordinates x', y', z' to a new set of coordinates x_1', x_2', x_3'. In particular case, they form an example of a linear or vector transformation equation of the given form

$$\left.\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned}\right\} \qquad (5)$$

Where a_{11}, a_{21}, a_{31} ..... are any set of constants.

The equation (5) can be written as

$$x_i' = \sum_{j=1}^{3} a_{ij}x_j \qquad (6)$$

Again we can write,

$$x_i' = \sum_{k=1}^{3} a_{ik}x_k \qquad (7)$$

We can write

$$\sum_{i=1}^{3} x_i'^2 = \sum_{i=1}^{3} x_i^2$$

$$\sum_{i=1}^{3} x_i'.x_i' = \sum_{i=1}^{3} x_i^2$$

From eqn 6 and 7 we get,

$$\sum_{i=1}^{3} x_i^2 = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} a_{ij}x_j\, a_{ik}x_k$$

$$= \sum_{j=1}^{3}\sum_{k=1}^{3}\left(\sum_{i=1}^{3} a_{ij}\, a_{ik}\right)x_jx_k$$

$$= \sum_{j,k=1}^{3} (\delta_{jk})x_jx_k \qquad (8)$$

Where $\delta_{jk} = \sum_{i=1}^{3} a_{ij}\, a_{ik}$

$$\delta_{jk} = 1;\ \ for\ i = j \qquad and \qquad \delta_{jk} = 0;\ \ for\ i \neq j$$

This is the required condition for orthogonality for the linear transformation.

Euler's Angles:

The **Euler angles** are three angles introduced by Leonhard Euler to describe the orientation of a rigid body with respect to a fixed coordinate system. They can also represent the orientation of a mobile frame of reference in physics or the orientation of a general basis in 3-dimensional linear algebra.

Euler angles can be defined by elemental geometry or by composition of rotations. The geometrical definition demonstrates that three composed *elemental rotations* (rotations about the axes of a coordinate system) are always sufficient to reach any target frame.

The three elemental rotations may be extrinsic (rotations about the axes *xyz* of the original coordinate system, which is assumed to remain motionless), or intrinsic (rotations about the axes of the rotating coordinate system *XYZ*, solidary with the moving body, which changes its orientation after each elemental rotation). Euler angles are typically denoted as α, β, γ, or φ, θ, ψ.

Without considering the possibility of using two different conventions for the definition of the rotation axes (intrinsic or extrinsic), there exist twelve possible sequences of rotation axes, divided in two groups:

- **Proper Euler angles** (*z*-*x*-*z*, *x*-*y*-*x*, *y*-*z*-*y*, *z*-*y*-*z*, *x*-*z*-*x*, *y*-*x*-*y*)
- **Tait–Bryan angles** (*x*-*y*-*z*, *y*-*z*-*x*, *z*-*x*-*y*, *x*-*z*-*y*, *z*-*y*-*x*, *y*-*x*-*z*).

Tait–Bryan angles are also called **Cardan angles**; **nautical angles**; **heading, elevation, and bank**; or **yaw, pitch, and roll**. Sometimes, both kinds of sequences are called "Euler angles". In that case, the sequences of the first group are called *proper* or *classic* Euler angles.

Euler's Angles:

Let us proceed to find three independent parameters which would completely specify the orientation of a rigid body - the so called Euler's angles.

Let x, y, z be the orthogonal space set of axes with **i**, **j** and **k** as unit vectors along these axes. Also x', y', z' be the orthogonal body set of axes with **i', j'** and **k'** unit vectors along these axes.

In order to account for the rotatory motion, we shall carry out the transformation from space set of axes to body set of axes. The transformation is worked out through three successive rotations performed in a certain specific order. That is, we rotate space set of axes x, y, z so as to coincide with body set of axes x', y', z', through three successive rotatory operations that are worked out one after another in a specific sequence.

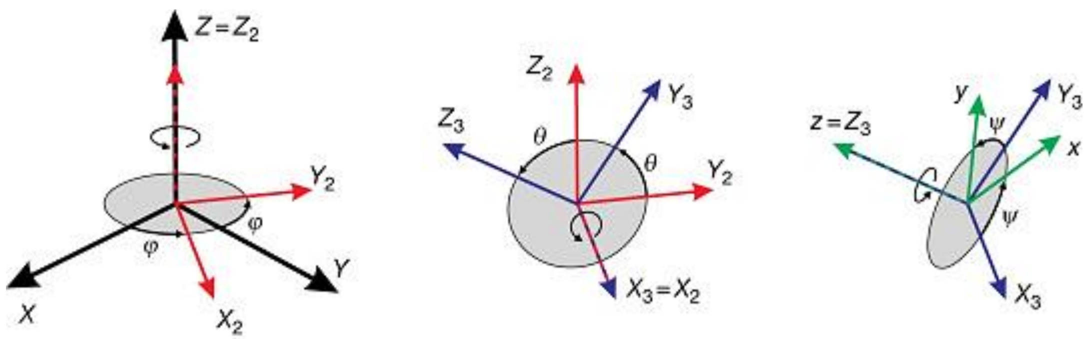


Fig. 1

First Rotation: First of all the space set of axes is rotated about the space z-axis so that the y-z plane takes new position y_1-z_1. This new plane contains the body Z-axis. The rotation angle is φ. The new axes are x_1 and y_1 with unit vectors $\boldsymbol{i}_1$ and $\boldsymbol{j}_1$ as shown in fig. 2. The transformation to this new set of axes x_1, y_1, z_1 from x, y, axes can be represented by the equations;

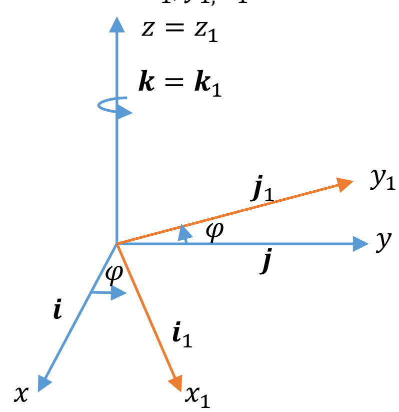


Fig. 2

$$\boldsymbol{i}_1 = \boldsymbol{i}\, cos\varphi + \boldsymbol{j}\, sin\varphi$$

$$\boldsymbol{j}_1 = -\boldsymbol{i}\, sin\varphi + \boldsymbol{j}\, cos\varphi$$

$$\boldsymbol{k}_1 = \boldsymbol{k}$$

Or,

$$\begin{bmatrix} \boldsymbol{i}_1 \\ \boldsymbol{j}_1 \\ \boldsymbol{k}_1 \end{bmatrix} = \begin{bmatrix} cos\varphi & sin\varphi & 0 \\ -sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{bmatrix}$$

the matrix of transformation is

$$D = \begin{bmatrix} cos\varphi & sin\varphi & 0 \\ -sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Second Rotation: Second rotation is performed about new x_1 axis so that z_1 axis coincides with body z' axis. This also brings the plane x2-y2 in the plane The axes x_2 and y_2 obtained after rotation about x1 through an angle θ still do not coincide with body x' and y' axes. The rotation angles is θ. The new axes y_2 and z_2 with unit vectors $\mathbf{j}_2$ and $\mathbf{k}_2$ are shown in fig. 3.

The transformation to this new set of axes $x_2, y_{2,} z_2$ from $x_1, y_{1,} z_1$ set of axes can represented by the equations

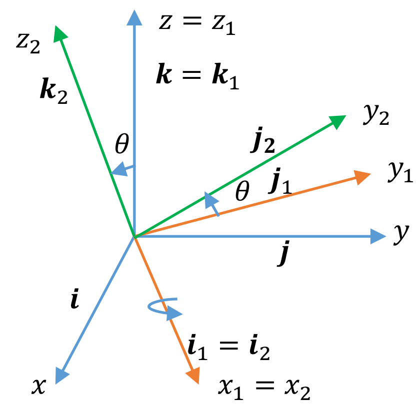
$$\boldsymbol{i}_2 = \boldsymbol{i}_1$$

$$\boldsymbol{j}_2 = \boldsymbol{j}_1 \, cos\theta + \boldsymbol{k}_1 \, sin\theta$$

$$\boldsymbol{k}_2 = -\boldsymbol{j}_1 \, sin\theta + \boldsymbol{k}_1 \, cos\theta$$

so that the matrix of transformation will be

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}$$



$Fig. 3$

As told earlier this transformation brings x_2y_2 plane into the plane x'y' of the body set of axes.

<u>**Third Rotation:**</u> Third rotation is performed about z_1 axis i.e., about z' axis so that the new axis x_3 coincides with body x' axis and the axis y_3 coincides with y' axis. This completes the transformation from space set of axes to body set of axes. The angle is ψ. The new axes are $y_3 (= y')$ and $x_3 (= x')$ as shown in fig. 4. Therefore i_3=i', j_3= j' and k_3= k'.

The transformation to this new set of axes $x_3, y_{3,} z_3$ which coincides with body set of axes x' y' z' can be represented by the equations:

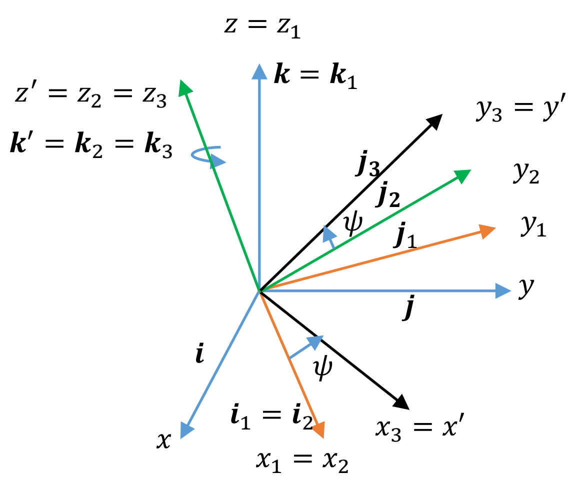
$$\boldsymbol{i}' = \boldsymbol{i}_2 \, cos\psi + \boldsymbol{j}_2 \, sin\psi$$

$$\boldsymbol{j}' = -\boldsymbol{i}_2 \, sin\psi + \boldsymbol{j}_2 \, cos\psi$$

$$\boldsymbol{k}' = \boldsymbol{k}_2$$

So that the transformation matrix is

$$B = \begin{bmatrix} cos\psi & sin\psi & 0 \\ -sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$Fig. 3$

Therefore, we have arrived at the body set of axes after three successive and sequential rotation of space set of axes. The complete matrix of transformation A will be

$$\begin{bmatrix} \boldsymbol{i}' \\ \boldsymbol{j}' \\ \boldsymbol{k}' \end{bmatrix} = \begin{bmatrix} \boldsymbol{i}_3 \\ \boldsymbol{j}_3 \\ \boldsymbol{k}_3 \end{bmatrix} = B \begin{bmatrix} \boldsymbol{i}_2 \\ \boldsymbol{j}_2 \\ \boldsymbol{k}_2 \end{bmatrix} = BC \begin{bmatrix} \boldsymbol{i}_1 \\ \boldsymbol{j}_1 \\ \boldsymbol{k}_1 \end{bmatrix} = BCD \begin{bmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{bmatrix}$$

Let, $A = BCD$

so that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

or the inverse transformation from body set of axes to space of axes will be given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

Thus the matrix of transformation A furnishing x' y' z'-axes from the space set of axes xyz directly is the product of matrices taken in the indicated order of rotations, viz. (φ, θ, ψ) or ABC :

A=BCD

$$A = \begin{bmatrix} cos\psi & sin\psi & 0 \\ -sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} cos\varphi & sin\varphi & 0 \\ -sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of orthogonal transformation matrix

1) <u>The value of determinant of the orthogonal matrix is invariant under similarity transformation</u>

In linear algebra, two n-by-n matrices A and B are called similar if there exists an invertible n-by-n matrix P such that

$$B = P^{-1}AP$$

Let A is an operator acting upon a vector F to produce a new vector G.

$$G = AF$$

If the coordinate system is transformed by a matrix B, the components of the vector G in the new system will be given by

$$BG = BAF$$

$$BG = BAB^{-1}BF \tag{1}$$

Equation 1 can be interpreted as the operator BAB^{-1} acting upon the vector F in the new coordinate system and produces a vector G in the new coordinate system. We may consider BAB^{-1} to be the form taken by the operator A when transformed to a new set of axis and given by

$$A' = BAB^{-1} \tag{2}$$

Any transformation of a matrix of the form of equation (2) is known as similarity transformation.

We can write from eqn 2

$$A'B = BAB^{-1}B$$

$$A'B = BA$$

This can be expressed in determinant form as

$$|A'||B| = |B||A|$$

Since the determinant of B is a number and not zero, so we divide above eqn. by $|B|$ and get,

$$|A'| = |A|$$

So, the value of the determinant of an orthogonal matrix is invariant under similarity transformation.

2) **Determinant of an orthogonal matrix is $|A| = \pm 1$**

As we know, the definition of matrix multiplication is identical with the multiplication of determinant, we can write

$$|AB| = |A|.|B|$$

Since the orthogonal condition of matrix is $\tilde{A}A = I$, we can write in determinant form

$$|\tilde{A}||A| = 1$$

We know that, the value of determinant is unaffected in interchanging any rows and column, we get,

$$|\tilde{A}|^2 = |A|^2 = 1$$

$$|\tilde{A}| = |A| = \pm 1$$

Which shows that the determinant of an orthogonal matrix can be +1 or -1.

3) **The orthogonality condition for the transformation matrix is**

$$\sum_{i=1}^{3} a_{ij}\, a_{ik} = \delta_{jk}$$

[Proved in previous]

Rate of Change of a Vector

Let us consider a reference frame xyz fixed in space and another reference frame x'y'z' fixed in body. At time t=0, their origin and base vectors are coincides. Let us now consider a point (x,y,z) in space, set of axes are describes by the position vector $\vec{\Omega}$ therefore the motion of P is described by $\vec{\Omega} = \vec{\Omega}(t)$.

At time t=0 we have,

$$\vec{\Omega} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \qquad \text{[in space set]}$$

$$\vec{\Omega} = x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}' \qquad \text{[in space set]}$$

Since x'y'z' axes rotates, the unit vectors $\hat{\imath}', \hat{\jmath}', \hat{k}'$ changes with respect to time, while the unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ remains constant. Since the point p moves, the velocity of p is different in the two system of axes for reason of their relative motion.

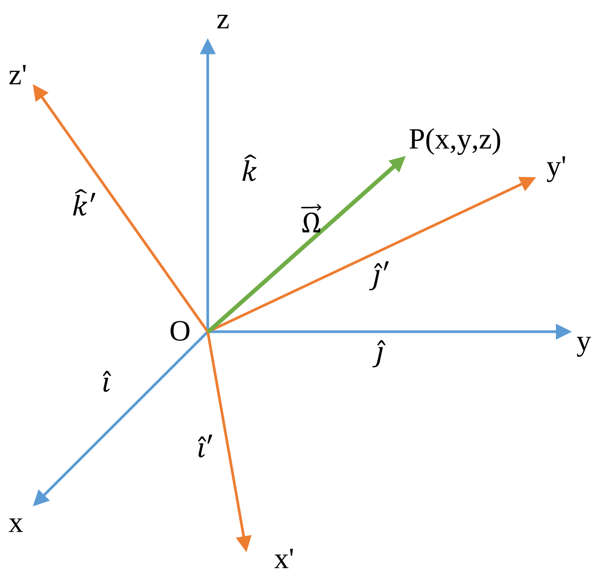
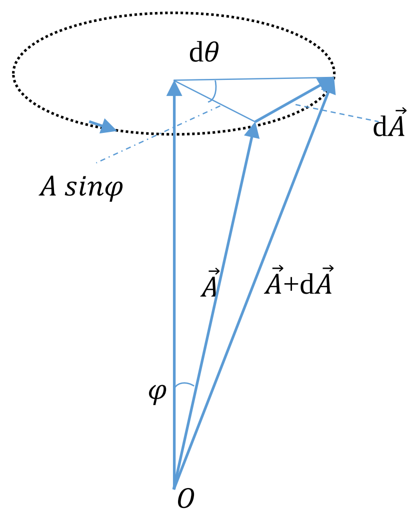


Fig. 1



$Fig. 2$

When a vector rotates then from fig.2 we can write,

$$\overrightarrow{dA} = \hat{\eta} d\theta A sin\varphi$$

$$= d\theta \hat{\eta} \times \vec{A}$$

Let us divide above eqn. by dt to find the rate of change of vector A is

$$\frac{d\vec{A}}{dt} = \frac{d\theta}{dt} \hat{\eta} \times \vec{A}$$

$$\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}$$

Where $\vec{\omega}$ is the angular velocity of the motion.

For unit vectors, we can write,

$$\frac{d\hat{\imath}'}{dt} = \vec{\omega} \times \hat{\imath}', \quad \frac{d\hat{\jmath}'}{dt} = \vec{\omega} \times \hat{\jmath}', \quad \frac{d\hat{k}'}{dt} = \vec{\omega} \times \hat{k}'$$

Therefore, the rate of change of vector $\vec{\Omega}$, when a body set of axes is rotating with angular velocity $\vec{\omega}$ with respect to space set of axes is

$$\frac{d\vec{\Omega}}{dt} = \frac{d}{dt}\left(x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}'\right)$$

$$= \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}' + \dot{z}'\hat{k}' + x'\frac{d\hat{\imath}'}{dt} + y'\frac{d\hat{\jmath}'}{dt} + z'\frac{d\hat{k}'}{dt}$$

$$= \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}' + \dot{z}'\hat{k}' + x'(\vec{\omega}\times\hat{\imath}') + y'(\vec{\omega}\times\hat{\jmath}') + z'(\vec{\omega}\times\hat{k}')$$

$$= \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}' + \dot{z}'\hat{k}' + \vec{\omega}\left(x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}'\right)$$

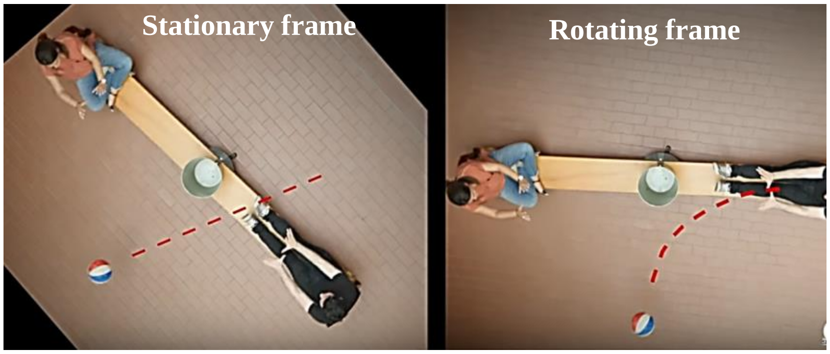
$$= \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}' + \dot{z}'\hat{k}' + \vec{\omega}\times\left(x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}'\right)$$

$$= \left(\frac{d\vec{\Omega}}{dt}\right)_{body} + \vec{\omega}\times\vec{\Omega}$$

$$\left(\frac{d\vec{\Omega}}{dt}\right)_{space} = \left(\frac{d\vec{\Omega}}{dt}\right)_{body} + \vec{\omega}\times\vec{\Omega}$$

Coriolis Force and Its Effect:

The Coriolis force is an inertial or fictitious force that acts on objects that are in motion within a frame of reference that rotates with respect to an inertial frame. In a reference frame with clockwise rotation, the force acts to the left of the motion of the object. In one with anticlockwise (or counterclockwise) rotation, the force acts to the right. Deflection of an object due to the Coriolis force is called the Coriolis effect.



The Coriolis Effect says that, when an observer is in rotating frame—whether it's on a playground toy or your home planet—objects moving in straight lines will appear to curve. This bizarre phenomenon affects many things, from the paths of missiles to the formation of hurricanes.

You may have heard that the Coriolis Effect makes water in the bathtub spiral down the drain in a certain way, or that it determines the way that a toilet flushes. That's actually wrong.

Although, as you may have noticed while tracking a hurricane on the news, storms in the Northern Hemisphere spin counterclockwise, while those in the Southern Hemisphere spin clockwise. Why do storms spin in different directions depending on their location? And why do they spin in the first place? The answer is the Coriolis Effect.

So, for an observer in a rotating frame the Coriolis effect seems to exert a very real force on objects. But there is no force. What the observer observes is just the result of his perspective.

When a body set of axes is rotating with constant angular velocity $\vec{\omega}$ with respect to the space set of axes then the rate of change of vector $\vec{r}$ is given by

$$\left(\frac{d\vec{r}}{dt}\right)_s = \left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega} \times \vec{r} \tag{1}$$

The operator equation will be

$$\left(\frac{d}{dt}\right)_s = \left(\frac{d}{dt}\right)_r + \vec{\omega} \tag{2}$$

Equation 1 can be written as

$$\vec{V}_s = \vec{V}_r + \vec{\omega} \times \vec{r} \tag{3}$$

Where $\vec{V}_s$ *and* $\vec{V}_r$ are the velocities of the particle relative to space and rotating system of axes of axes respectively.

Now, equation (2) can be used to obtain the time rate of change of $(\vec{V})_{space}$ as

$$\left(\frac{d\vec{V}_s}{dt}\right)_s = \left(\frac{d\vec{V}_s}{dt}\right)_r + \vec{\omega} \times \vec{V}_s$$

$$\vec{a}_s = \frac{d}{dt}\left(\vec{V}_r + \vec{\omega} \times \vec{r}\right)_r + \vec{\omega} \times \left(\vec{V}_r + \vec{\omega} \times \vec{r}\right)$$

$$\vec{a}_s = \left(\frac{d\vec{V}_r}{dt}\right)_r + \left(\frac{d\vec{\omega}}{dt}\right)_r \times \vec{r} + \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega} \times \vec{V}_r + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\vec{a}_s = (\vec{a})_r + 0 + \vec{\omega} \times \left(\vec{V}\right)_r + \vec{\omega} \times \vec{V}_r + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\vec{a}_s = \vec{a}_r + 0 + 2\left(\vec{\omega} \times \vec{V}_r\right) + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Multiplying both sides of equation 4 by the mass of the particle, m, we get,

$$m\vec{a}_s = \mathrm{m}\vec{a}_r + 2m\left(\vec{\omega} \times \vec{V}_r\right) + m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\vec{F} = \vec{F}_{eff} + 2m\left(\vec{\omega} \times \vec{V}_r\right) + m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\vec{F}_{eff} = \vec{F} - 2m\left(\vec{\omega} \times \vec{V}_r\right) - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

The second term is the so-called Coriolis force and the last term is the (generalized) "centrifugal force". The latter leads to a radially outward pointing force (counteracting gravity) plus a force towards the equatorial plane for any position at higher (positive or negative) latitudes. Of course, at the poles these two forces cancel which is not surprising, as the rotation does not actually contribute to any motion there. The Coriolis force gives a push "sideways" to an object moving in any direction other than Earth's axis. In particular, an object moving on Earth's surface at any point other than the Equator will have a component of its velocity perpendicular to **ω** and thus be pushed West if it moves North on the Northern Hemisphere, South if it moves West etc. This means that missiles will be missing the mark they are aimed at, and a stream of fluid (air, ocean water, etc.) will be bend into a circular pattern (counterclockwise in the Northern Hemisphere, clockwise in the southern one). Hence the regular patterns of hurricanes, ocean currents etc.

Euler's Equation of Motion of a Rigid Body:

Let us consider a rigid body rotating about a certain axes except three space axes and whose axes are chosen as principal axes having direction of the unit vectors $\vec{e}_1, \vec{e}_2, and\ \vec{e}_3$ respectively. Then the angular momentum becomes

$$\vec{\Omega} = I_1\omega_1\vec{e}_1 + I_2\omega_2\,\vec{e}_2 + I_3\omega_3\,\vec{e}_3 \quad (1)$$

The angular velocity becomes

$$\vec{\omega} = \omega_1\vec{e}_1 + \omega_2\,\vec{e}_2 + \omega_3\,\vec{e}_3$$

If 's' and b refer to the space (fixed) and body (moving) axes respectively then we have (2)

$$\left(\frac{d\vec{\Omega}}{dt}\right)_s = \left(\frac{d\vec{\Omega}}{dt}\right)_b + \vec{\omega}\times\vec{\Omega}$$

$$= \frac{d}{dt}(I_1\omega_1\vec{e}_1 + I_2\omega_2\,\vec{e}_2 + I_3\omega_3\,\vec{e}_3) + (\omega_1\vec{e}_1 + \omega_2\,\vec{e}_2 + \omega_3\,\vec{e}_3)\times(I_1\omega_1\vec{e}_1 + I_2\omega_2\,\vec{e}_2 + I_3\omega_3\,\vec{e}_3)$$

$$= I_1\dot{\omega}_1\vec{e}_1 + I_2\dot{\omega}_2\,\vec{e}_2 + I_3\dot{\omega}_3\,\vec{e}_3 + (\omega_1\vec{e}_1 + \omega_2\,\vec{e}_2 + \omega_3\,\vec{e}_3)\times(I_1\omega_1\vec{e}_1 + I_2\omega_2\,\vec{e}_2 + I_3\omega_3\,\vec{e}_3)$$

$$= I_1\dot{\omega}_1\vec{e}_1 + I_2\dot{\omega}_2\,\vec{e}_2 + I_3\dot{\omega}_3\,\vec{e}_3 + \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{bmatrix}$$

$$= I_1\dot{\omega}_1\vec{e}_1 + I_2\dot{\omega}_2\,\vec{e}_2 + I_3\dot{\omega}_3\,\vec{e}_3 + \vec{e}_1(I_3\omega_3\omega_2 - I_2\omega_2\omega_3) + \vec{e}_2(\omega_3 I_1\omega_1 - I_3\omega_3\omega_1) + \vec{e}_3(\omega_1 I_2\omega_2 - I_1\omega_1\omega_2)$$

$$= \vec{e}_1\{I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2)\} + \vec{e}_2\{I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3)\} + \vec{e}_3\{I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1)\} \quad (3)$$

As we know the rate of change of angular momentum is torque then we can write,

$$\vec{\Lambda} = \left(\frac{d\vec{\Omega}}{dt}\right)_s$$

Where we can express the external torque $\vec{\Lambda}$ by

$$\vec{\Lambda} = \Lambda_1\vec{e}_1 + \Lambda_2\vec{e}_2 + \Lambda_3\vec{e}_3$$

Where $\Lambda_1, \Lambda_2, \Lambda_3$ are the component of external torque along principal axes.

We can write equation 3 as

$$\Lambda_1\vec{e}_1 + \Lambda_2\vec{e}_2 + \Lambda_3\vec{e}_3 = \vec{e}_1\{I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2)\} + \vec{e}_2\{I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3)\} + \vec{e}_3\{I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1)\}$$

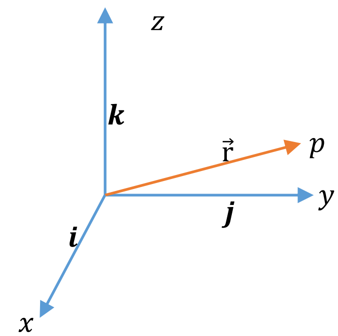
Equating the coefficients we can write

$$\left.\begin{aligned} \Lambda_1 &= I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) \\ \Lambda_2 &= I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) \\ \Lambda_3 &= I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) \end{aligned}\right\} \quad (4)$$

These are the Euler's equations of motion in component form.

Angular momentum, moment of inertia and angular velocity of rigid body:

Let us consider a fixed xyz coordinate system having origin O. If we consider a particle p moving with angular velocity $\vec{\omega}$ and having angular momentum $\vec{\Omega}$ with fixed origin O, we can express the position vector, angular velocity and angular momentum of this particle as



$Fig. 1$

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\vec{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}$$

$$\vec{\Omega} = \Omega_x\hat{\imath} + \Omega_y\hat{\imath} + \Omega_z\hat{\imath}$$

respectively.

Then the angular momentum is given by

$$\vec{\Omega} = \sum_{i=1}^{N} \vec{r}_i \times \vec{P}_i$$

$$\vec{\Omega} = \sum_{i=1}^{N} m_i(\vec{r}_i \times \vec{v}_i) = \sum_{i=1}^{N} m_i(\vec{r}_i \times \vec{\omega} \times \vec{r}_i)$$

From the property of vector algebra we can write,

$$\vec{\Omega} = \sum_{i=1}^{N} m_i(\vec{r}_i \times \vec{\omega} \times \vec{r}_i) = \sum_{i=1}^{N} m_i\big(\vec{\omega}\, r_i^2 - (\vec{r}_i.\vec{\omega})\vec{r}_i\big)$$

In component form we can write,

$$\Omega_x\hat{\imath} + \Omega_y\hat{\imath} + \Omega_z\hat{\imath} = \sum_{i=1}^{N} m_i\left((\omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k})(x_i{}^2 + y_i{}^2 + z_i{}^2) - (\omega_x x_i + \omega_y y_i + \omega_z z_i)(x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k})\right)$$

$$= \sum_{i=1}^{N} m_i\Big(\cancel{\omega_x x_i{}^2\hat{\imath}} + \omega_x y_i{}^2\hat{\imath} + \omega_x z_i{}^2\hat{\imath} + \omega_y x_i{}^2\hat{\jmath} + \cancel{\omega_y y_i{}^2\hat{\jmath}} + \omega_y z_i{}^2\hat{\jmath} + \omega_z x_i{}^2\hat{k} + \omega_z y_i{}^2\hat{k} + \cancel{\omega_z z_i{}^2\hat{k}}$$
$$- \big(\cancel{\omega_x x_i{}^2\hat{\imath}} + \omega_x x_i y_i\hat{\jmath} + \omega_x x_i z_i\hat{k} + \omega_y y_i x_i\hat{\imath} + \cancel{\omega_y y_i{}^2\hat{\jmath}} + \omega_y y_i z_i\hat{k} + \omega_z z_i x_i\hat{\imath} + \omega_z z_i y_i\hat{\jmath} + \cancel{\omega_z z_i{}^2\hat{k}}\big)\Big)$$

$$= \left[\omega_x\sum_{i=1}^{N} m_i(y_i{}^2 + z_i{}^2) - \omega_y\sum_{i=1}^{N} m_i y_i x_i - \omega_z\sum_{i=1}^{N} m_i z_i x_i\right]\hat{\imath}$$
$$+ \left[-\omega_x\sum_{i=1}^{N} m_i x_i y_i + \omega_y\sum_{i=1}^{N} m_i\left(x_i{}^2 + z_i{}^2\right) - \omega_z\sum_{i=1}^{N} m_i z_i y_i\right]\hat{\jmath}$$
$$+ \left[-\omega_x\sum_{i=1}^{N} m_i x_i z_i - \omega_y\sum_{i=1}^{N} m_i y_i z_i + \omega_z\sum_{i=1}^{N} m_i(x_i{}^2 + y_i{}^2)\right]\hat{k}$$

$$= \left[\omega_x I_{xx} + \omega_y I_{xy} + \omega_z I_{xz}\right]\hat{\imath} + \left[\omega_x I_{yx} + \omega_y I_{yy} + \omega_z I_{yz}\right]\hat{\jmath} + \left[\omega_x I_{zx} + \omega_y I_{zy} + \omega_z I_{zz}\right]\hat{k}$$

Equating the coefficients of unit vectors we get

$$\Omega_x = \omega_x I_{xx} + \omega_y I_{xy} + \omega_z I_{xz}$$

$$\Omega_y = \omega_x I_{yx} + \omega_y I_{yy} + \omega_z I_{yz}$$

$$\Omega_z = \omega_x I_{zx} + \omega_y I_{zy} + \omega_z I_{zz}$$

Where

$$I_{xx} = \sum_{i=1}^{N} m_i({y_i}^2 + {z_i}^2) \qquad I_{yy} = \sum_{i=1}^{N} m_i\left({x_i}^2 + {z_i}^2\right) \qquad I_{zz} = \sum_{i=1}^{N} m_i({x_i}^2 + {y_i}^2)$$

$$I_{xy} = -\sum_{i=1}^{N} m_i y_i x_i = I_{yx}$$

$$I_{yz} = -\sum_{i=1}^{N} m_i y_i z_i = I_{zy}$$

$$I_{zx} = -\sum_{i=1}^{N} m_i x_i z_i = I_{xz}$$

The quantities I_{xx}, I_{yy}, I_{zz} are the moment of inertia found from perpendicular axis theorem and I_{xy}, I_{yz}, I_{zx} are the product of moment of inertia about x,y,z axes respectively. Above equation can be written in the matrix form as

$$\begin{bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz}\\ I_{yx} & I_{yy} & I_{yz}\\ I_{zx} & I_{zy} & I_{zz}\end{bmatrix}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}$$

Centroidal axis *is any line that will pass through the centroid of the cross section. There may be infinite number of lines passing through the point of center of gravity. Therefore, a cross section has an infinite number of centroidal axes. Two axes out of these infinite directions are important which are termed as principal axis. They are 1) major principal axis and 2) minor principal axis.*

Major Principal Axis:

It is a centroidal axis about which the moment of inertia is the largest compared with the values among the other axes. The strongest axis of any cross section is called major principal axis.

Minor Principal Axis:

It is a centroidal axis about which the moment of inertia is the smallest compared with the values among the other axes. The minor principal axis is also called weakest axis. The failure due to bending starts and prolongs from weakest axis of any cross section.

The ***neutral axis*** *is an axis at which the strain is zero under the application of bending forces on a bending element. If the stresses are within the yield limit and linear in nature, neutral axis will coincide with the axis of centroid.*

Force free motion of a rigid body:

The first point that we can make is that, provided that no external torques act on the body, its angular momentum L is constant in magnitude and direction. A second point is that, provided the body is rigid and has no internal degrees of freedom, the rotational kinetic energy T is constant.

Let us consider a rigid body which is symmetric about an axis has one point fixed on it. Assuming that there is no forces acting than reaction force at the fixed point. Now Euler's equation of motion for the force free motion becomes,

$$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = 0$$

$$I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = 0$$

$$I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = 0$$

Let the axes of symmetry coincide with one of the principle axes, say one having direction $\vec{e}_3$. Then $I_1 = I_2$, so that Euler's equation of motion becomes.

$$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = 0 \quad (1)$$

$$I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = 0 \quad (2)$$

$$I_3\dot{\omega}_3 = 0 \quad (3)$$

From eqn. 3 we get,

$$\int I_3\dot{\omega}_3\,dt = 0$$

$$\omega_3 = constant = A$$

Now, dividing eqn. 1 and 2 by $I_1 = I_2$ we get

$$\dot{\omega}_1 + \omega_2\omega_3\left(\frac{I_3 - I_2}{I_1}\right) = 0 \quad (4)$$

$$\dot{\omega}_2 + \omega_1\omega_3\left(\frac{I_1 - I_3}{I_2}\right) = 0 \quad (5)$$

Substituting the value of $\omega_3 = A$, above two equations becomes

$$\dot{\omega}_1 + \omega_2 A\left(\frac{I_3 - I_2}{I_1}\right) = 0 \quad (6)$$

$$\dot{\omega}_2 + \omega_1 A\left(\frac{I_1 - I_3}{I_2}\right) = 0 \qquad (7)$$

Differentiating eqn. 7 with respect to time we get

$$\ddot{\omega}_2 + \dot{\omega}_1 A\left(\frac{I_1 - I_3}{I_2}\right) = 0$$

Putting the value of $\dot{\omega}_1$ from eqn. 6,

$$\ddot{\omega}_2 - A\left(\frac{I_1 - I_3}{I_2}\right)\omega_2 A\left(\frac{I_3 - I_2}{I_1}\right) = 0$$

$$\ddot{\omega}_2 - \omega_2 A^2\left(\frac{I_1 - I_3}{I_2}\right)^2 = 0$$

$$\ddot{\omega}_2 + \omega_2 k^2 = 0 \qquad \text{Where} \quad k = -A\left(\frac{I_1 - I_3}{I_2}\right)$$

Solution of this equation is

$$\omega_2 = B\cos kt + C\sin kt$$

When t=0, there is no component of angular velocity so $\omega_2 = 0$ and then we get B=0, so

$$\omega_2 = C\sin kt$$

$$\dot{\omega}_2 = Ck\cos kt$$

Then from eqn. 7 we get

$$Ck\cos kt - \omega_1 k = 0$$

$$\omega_1 = C\cos kt$$

Then the angular velocity becomes

$$\vec{\omega} = C\cos kt\,\vec{e}_1 + C\sin kt\,\vec{e}_2 + A\,\vec{e}_3$$

Magnitude of this angular velocity is

$$\omega^2 = C^2\cos^2 kt + C^2\sin^2 kt + A^2$$

$$\omega = \sqrt{C^2 + A^2}$$

Which is constant and can be the angular frequency of earth.

Again the precessional frequency of the rigid body about z-axis is $f = \frac{k}{2\pi}$

$$f = A\left(\frac{I_1 - I_3}{2\pi I_2}\right) = A\left(\frac{I_1 - I_3}{2\pi I_1}\right)$$

Equation of motion of a spinning top

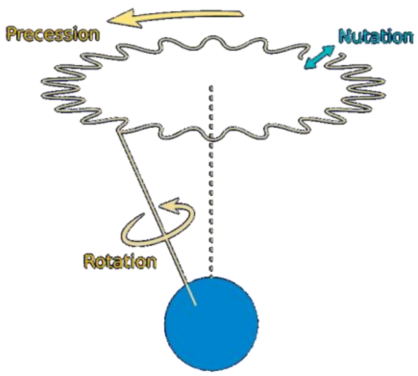


Fig. 1

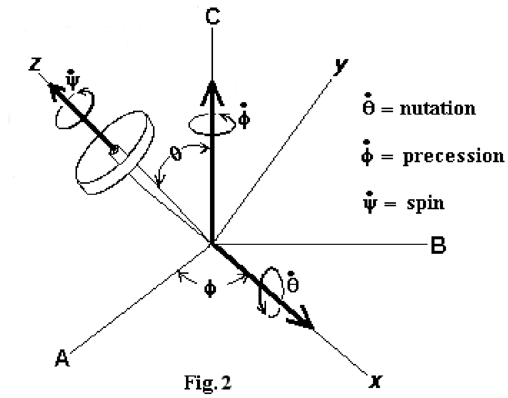


Fig. 2

Let XYZ represents the fixed set of axes having origin O. Let X'Y'Z' represents the principal axis of the top having the same origin. If we choose the rotation of the XY plane, the OZ, OZ' and OY' are coplanar. The X' axis lie in the X'Y' plane. We take the body z'-axis is the symmetry axis so $I_1 = I_2$.

Let the line ON makes an angle ψ with the X' axis, which is assumed to be attached with the top. The angular velocity corresponding to the rotation of the X'Y'Z' axes with respect to the XYZ axis is

$$\vec{\omega} = \omega_1 \vec{e}_1 + \omega_2\, \vec{e}_2 + \omega_3\, \vec{e}_3 \tag{1}$$

Since the top is spinning about the Z' axis, in order to obtain angular momentum, we must use the fact that in addition to the component ω_3 due to the rotation of the X'Y'Z' axes we must also have the component $\vec{s} = \omega_3\, \vec{e}_3 = \dot{\psi}\vec{e}_3$.

$$\vec{\Omega} = I_1\omega_1\vec{e}_1 + I_2\omega_2\, \vec{e}_2 + I_3(\omega_3 + s)\, \vec{e}_3 \tag{2}$$

If 's' and b refer to the space (fixed) and body (moving) axes respectively then we have

$$\left(\frac{d\vec{\Omega}}{dt}\right)_s = \left(\frac{d\vec{\Omega}}{dt}\right)_b + \vec{\omega} \times \vec{\Omega} \tag{3}$$

$$= \frac{d}{dt}\{I_1\omega_1\vec{e}_1 + I_2\omega_2\, \vec{e}_2 + I_3(\omega_3 + s)\, \vec{e}_3\} + (\omega_1\vec{e}_1 + \omega_2\, \vec{e}_2 + \omega_3\, \vec{e}_3) \times (I_1\omega_1\vec{e}_1 + I_2\omega_2\, \vec{e}_2 + I_3(\omega_3 + s)\, \vec{e}_3)$$

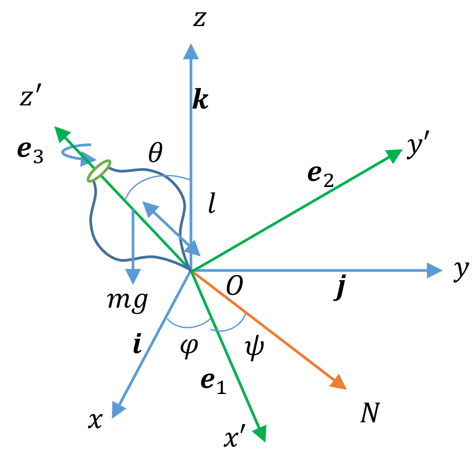
$$= I_1\dot{\omega}_1\vec{e}_1 + I_2\dot{\omega}_2\, \vec{e}_2 + I_3(\dot{\omega}_3 + \dot{s})\, \vec{e}_3 + (\omega_1\vec{e}_1 + \omega_2\, \vec{e}_2 + \omega_3\, \vec{e}_3) \times \{I_1\omega_1\vec{e}_1 + I_2\omega_2\, \vec{e}_2 + I_3(\omega_3 + s)\vec{e}_3\}$$

$$= I_1\dot{\omega}_1\vec{e}_1 + I_2\dot{\omega}_2\, \vec{e}_2 + I_3(\dot{\omega}_3 + \dot{s})\, \vec{e}_3 + \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3(\omega_3 + s) \end{bmatrix}$$

$$= I_1\dot{\omega}_1\vec{e}_1 + I_2\dot{\omega}_2\, \vec{e}_2 + I_3(\dot{\omega}_3 + \dot{s})\, \vec{e}_3 + \vec{e}_1\{\omega_2 I_3(\omega_3 + s) - I_2\omega_2\omega_3\} + \vec{e}_2\{\omega_3 I_1\omega_1 - I_3(\omega_3 + s)\omega_1\} + \vec{e}_3(\omega_1 I_2\omega_2 - I_1\omega_1\omega_2)$$

$$= \vec{e}_1\{I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) + \omega_2 I_3 s\} + \vec{e}_2\{I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) - I_3\omega_1 s\} + \vec{e}_3\{I_3(\dot{\omega}_3 + \dot{s}) + \omega_1\omega_2(I_2 - I_1)\} \quad (4)$$

Now the total torque about O is



$Fig. 3$

$$\vec{\Lambda} = -(l\vec{e}_3) \times (mg\hat{k})$$

$$= -mgl(\vec{e}_3 \times \hat{k})$$

$$= -mgl[\vec{e}_3 \times \{(\hat{k}.\vec{e}_1)\vec{e}_1 + (\hat{k}.\vec{e}_2)\vec{e}_2 + (\hat{k}.\vec{e}_3)\vec{e}_3\}]$$

$$= -mgl[\vec{e}_3 \times \{0 + \cos(90 - \theta)\,\vec{e}_2 + \cos\theta\,\vec{e}_3\}]$$

$$= -mgl[\vec{e}_3 \times \{\sin\theta\,\vec{e}_2 + \cos\theta\,\vec{e}_3\}]$$

$$= -mgl\sin\theta\,[\vec{e}_3 \times \vec{e}_2]$$

$$= -mgl\sin\theta\,[-\vec{e}_1] = mgl\sin\theta\,\vec{e}_1$$

As we know the torque $\vec{\Lambda} = \left(\frac{d\vec{\Omega}}{dt}\right)_s$ then from eqn. 4 we get,

$$mgl\sin\theta\,\vec{e}_1 = \vec{e}_1\{I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) + \omega_2 I_3 s\} + \vec{e}_2\{I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) - I_3\omega_1 s\} + \vec{e}_3\{I_3(\dot{\omega}_3 + \dot{s}) + \omega_1\omega_2(I_2 - I_1)\}$$

Equating the coefficients of unit vectors we get,

$$\left.\begin{aligned} I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) + \omega_2 I_3 s &= mgl\sin\theta \\ I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) - I_3\omega_1 s &= 0 \\ I_3(\dot{\omega}_3 + \dot{s}) + \omega_1\omega_2(I_2 - I_1) &= 0 \end{aligned}\right\} \quad (5)$$

For symmetry consideration let us put $I_2 = I_1$ in above equation,

$$\left.\begin{aligned} I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) + \omega_2 I_3 s &= mgl\sin\theta \\ I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) - I_3\omega_1 s &= 0 \\ I_3(\dot{\omega}_3 + \dot{s}) &= 0 \end{aligned}\right\} \quad (6)$$

Above equations are known as the equation of motion of a spinning top.

In order to express these equations in terms of the Euler's angle we have to express $\omega_1, \omega_2,$ *and* ω_3 in terms of Euler's angle.

Now let us express the components of $\vec{\omega}$ in terms of Euler's angles

$$\vec{\omega} = \omega_\varphi\vec{k} + \omega_\theta\vec{i}_2 + \omega_\psi\vec{e}_3$$

Where

$\vec{k} = \sin\theta\sin\psi\ \vec{e}_1 + \cos\psi\sin\theta\,\vec{e}_2 + \cos\theta\ \vec{e}_3$

$\vec{\imath}_2 = \cos\psi\ \vec{e}_1 - \sin\psi\,\vec{e}_2$

$\vec{e}_3 = \vec{e}_3$

Then we can write

$\vec{\omega} = \omega_\varphi(\sin\theta\sin\psi\ \vec{e}_1 + \cos\psi\sin\theta\,\vec{e}_2 + \cos\theta\ \vec{e}_3) + \omega_\theta(\cos\psi\ \vec{e}_1 - \sin\psi\,\vec{e}_2) + \omega_\psi\vec{e}_3$

$= \dot{\varphi}(\sin\theta\sin\psi\ \vec{e}_1 + \cos\psi\sin\theta\,\vec{e}_2 + \cos\theta\ \vec{e}_3) + \dot{\theta}(\cos\psi\ \vec{e}_1 - \sin\psi\,\vec{e}_2) + \dot{\psi}\vec{e}_3$

$= (\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\,)\vec{e}_1 + (\dot{\varphi}\cos\psi\sin\theta - \dot{\theta}\sin\psi)\vec{e}_2 + (\dot{\varphi}\cos\theta\ + \dot{\psi})\vec{e}_3$

So the components are

$\omega_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$

$\omega_2 = \dot{\varphi}\cos\psi\sin\theta - \dot{\theta}\sin\psi$

$\omega_3 = \dot{\varphi}\cos\theta\ + \dot{\psi}$

Putting $\psi = 0$, we get, $\omega_1 = \dot{\theta}$, $\omega_2 = \dot{\varphi}\sin\theta$ and $\omega_3 = \dot{\varphi}\cos\theta$

Then equation 6 becomes

$$\left.\begin{aligned} &I_1\ddot{\theta} + (I_3 - I_2)\dot{\varphi}^2\sin\theta\cos\theta + I_3 s\dot{\varphi}\sin\theta = mgl\sin\theta \\ &I_2(\ddot{\varphi}\sin\theta + \dot{\varphi}\cos\theta) + (I_1 - I_3)\dot{\theta}\dot{\varphi}\cos\theta - I_3\dot{\theta}s = 0 \\ &I_3(\ddot{\varphi}\cos\theta - \dot{\varphi}\sin\theta + \dot{s}) = 0 \end{aligned}\right\} \qquad (7)$$

These are required equation of motion of a symmetric top.

Kinetic Energy of Motion of a Spinning Top

[Give small description from previous point (including figure).]
Considering symmetry we can write the kinetic energy of the symmetric top

$$T = \frac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2$$

Putting the value of $\omega_1, \omega_2,\ and\ \omega_3$ we get,

$$T = \frac{1}{2}I_1\left\{(\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\,)^2 + (\dot{\varphi}\cos\psi\sin\theta - \dot{\theta}\sin\psi)^2\right\} + \frac{1}{2}I_3(\dot{\varphi}\cos\theta\ + \dot{\psi})^2$$

$$= \frac{1}{2}I_1\{\dot{\varphi}^2\sin^2\theta\sin^2\psi + 2\dot{\varphi}\sin\theta\sin\psi\,\dot{\theta}\cos\psi\ + \dot{\theta}^2\cos^2\psi\ + \dot{\varphi}^2\cos^2\psi\sin^2\theta$$
$$- 2\,\dot{\varphi}\cos\psi\sin\theta\,\dot{\theta}\sin\psi + \dot{\theta}^2\sin^2\psi\} + \frac{1}{2}I_3(\dot{\varphi}\cos\theta\ + \dot{\psi})^2$$

$$= \frac{1}{2} I_1 \{ \dot{\varphi}^2 \sin^2\theta \sin^2\psi + \dot{\theta}^2 \cos^2\psi \; + \dot{\varphi}^2 \cos^2\psi \sin^2\theta + \dot{\theta}^2 \sin^2\psi \} + \frac{1}{2} I_3 (\dot{\varphi} \cos\theta \; + \dot{\psi})^2$$

$$= \frac{1}{2} I_1 \{ \dot{\varphi}^2 \sin^2\theta \, (\sin^2\psi + \cos^2\psi) + \dot{\theta}^2 (\cos^2\psi \; + \sin^2\psi) \} + \frac{1}{2} I_3 (\dot{\varphi} \cos\theta \; + \dot{\psi})^2$$

$$= \frac{1}{2} I_1 \{ \dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2 \} + \frac{1}{2} I_3 (\dot{\varphi} \cos\theta \; + \dot{\psi})^2$$

This is the required equation for kinetic energy of a symmetric top in terms of Euler's angle.