PH501 · ADVANCED NUCLEAR PHYSICS

Lecture 2 — Reaction kinematics

M.J. Kobra, Department of Physics, University of Rajshahi

1. Setup and goal

We have a reaction $A(a, b)B$: a projectile a (kinetic energy T_a) hits a target A at rest, and out comes an ejectile b at lab angle θ_L (energy T_b) plus a recoiling residual B. The energetics are fixed by the Q-value,

$$Q = (m_a + m_A - m_b - m_B)\, c^2$$

with positive Q if energy is released and negative Q if it has to be supplied. The whole question for this lecture is short: given T_a, θ_L, the masses and Q, what is T_b? Nothing fancier than energy and momentum conservation is needed. The work all goes into moving between the lab (LAB) and centre-of-mass (CM) frames, and it ends in two formulae: equation (8) for T_b at any angle, and equation (9) for the threshold of an endothermic reaction.

2. LAB and CM frames

We work in two frames. The LAB is where the target sits at rest and where the detectors live, so every measured angle and energy is a LAB quantity. The CM is the frame in which the total momentum is zero; it is the natural frame for the algebra because the two-body problem is symmetric there.

The CM moves through the LAB at some velocity v_{CM}. Writing the total CM momentum as zero gives $m_a\, v_a^* + m_A\, v_A^* = 0$ (stars denote CM quantities), and the Galilean relation $v = v^* + v_{CM}$, with $v_A = 0$ in the LAB, fixes $v_A^* = -v_{CM}$. Putting the two together gives the CM velocity,

$$v_{CM} = m_a\, v_a \,/\, (m_a + m_A) \tag{1}$$

Two limits are worth keeping in mind. For a heavy target ($m_A \to \infty$) it vanishes, $v_{CM} \to 0$, so the LAB and CM frames coincide. For equal masses ($m_A = m_a$, as in p-p scattering) the CM carries half the projectile velocity, $v_{CM} = ½\, v_a$.

3. Splitting the energy

The beam energy T_a divides cleanly into the bulk motion of the CM through the LAB plus the relative motion inside the CM:

$$T_a = ½\, M\, v_{CM}^2 + T_{CM}, \qquad M = m_a + m_A \tag{2}$$

The first piece, $½\, M\, v_{CM}^2$, is the bulk motion. Momentum conservation keeps the CM moving at v_{CM} throughout the reaction, so this energy is locked away and cannot go into rest mass, excitation, or extra particles. The second piece, T_{CM}, is what is actually available to the reaction. Substituting (1) into (2) and solving for it gives

$$T_{CM} = T_a \cdot m_A \,/\, (m_a + m_A) \tag{3}$$

For ^{40}Ca(d, p) at $T_d = 10$ MeV this leaves $T_{CM} = 10 \times 40/42 \approx 9.52$ MeV available, with about 0.48 MeV tied up in CM motion. (For equal masses only half of T_a would ever be available.)

4. The Q-value equation

The derivation is short. Energy conservation says the kinetic energy afterwards is larger by exactly Q,

$$T_a + Q = T_b + T_B \tag{4}$$

Momentum conservation, with the beam along z, is $p_a = p_b + p_B$. Writing $p_B = p_a - p_b$ and squaring gives

$$p_B^2 = p_a^2 + p_b^2 - 2\, p_a\, p_b \cos\theta_L \tag{6}$$

The minus sign is just from squaring a vector difference, but it is what brings the angle θ_L into the answer. Now put $p^2 = 2mT$ in for each particle (and $p_a p_b = 2\sqrt{(m_a m_b T_a T_b)}$) to rewrite this in terms of energies,

$$m_B\, T_B = m_a\, T_a + m_b\, T_b - 2\sqrt{(m_a\, m_b\, T_a\, T_b)} \cos\theta_L \tag{7}$$

We never measure T_B, so get rid of it with (4), $T_B = T_a + Q - T_b$. Rearranging and dividing by m_B leaves the Q-value equation,

$$Q = T_b(1 + m_b/m_B) - T_a(1 - m_a/m_B) - (2/m_B)\sqrt{(m_a\, m_b\, T_a\, T_b)} \cos\theta_L \tag{8}$$

It is a quadratic in $x \equiv \sqrt{T_b}$, namely $ax^2 + bx + c = 0$ with $a = 1 + m_b/m_B$, $b = -(2/m_B)\sqrt{(m_a\, m_b\, T_a)} \cos\theta_L$ and $c = -[Q + T_a(1 - m_a/m_B)]$; take the positive root. (Satchler does this in CM variables — Appendix B, eq. B7.)

5. Worked example: $^{40}Ca(d, p)^{41}Ca$ at $\theta_L = 30°$

Take $T_d = 10$ MeV, $Q = +6.14$ MeV and $\theta_L = 30°$ (so $\cos\theta_L = 0.866$), with masses $m_d = 2$, $m_p = 1$, $m(^{40}Ca) = 40$ and $m(^{41}Ca) = 41$ u. Substituting into (8),

$$6.14 = T_p(1 + 1/41) - 10(1 - 2/41) - (2/41)\sqrt{(2·1·10·T_p)}\,(0.866)$$

Evaluating the three coefficients (1.0244, 9.512 and 0.0844) and writing $x = \sqrt{T_p}$ reduces this to a quadratic, which we solve for the positive root:

$$1.0244\, x^2 - 0.0844\, x - 15.652 = 0 \quad\rightarrow\quad x = 3.951 \quad\rightarrow\quad T_p = x^2 \approx 15.6 \text{ MeV}$$

As a check, energy conservation gives $T_B = 10 + 6.14 - 15.6 \approx 0.54$ MeV, the same order as the $T_p·m_p/m_B \approx 0.38$ MeV expected from momentum balance. Notice that T_p comes out larger than the beam energy T_d: because $Q > 0$, rest mass is converted into kinetic energy, and the lighter proton walks off with most of it.

6. Kinematic locus and spectroscopy

Reading (8) as T_b versus θ_L at fixed beam energy and Q traces out the *kinematic locus*. For our ground-state case it runs from about 16 MeV at 0° down to about 14 MeV at 180°, the forward peak simply reflecting the beam momentum pointing forward.

If the residual is instead left in an excited state of energy E_x, energy conservation becomes $T_a + Q = T_b + T_B + E_x$, which is the same as using an effective $Q_{eff} = Q - E_x$ in (8); the whole locus simply drops by about E_x. So at a single fixed angle each final state of B shows up as a separate peak in the ejectile spectrum, and the level scheme can be read straight off the peak positions (Table 1). Those positions are pure kinematics. The peak *heights* are a different story, set by the angular-momentum transfer ℓ, the spectroscopic factor S and the DWBA matrix element, which we come to in Lectures 6 and 8.

T_p (MeV)	^{41}Ca state	E_x (MeV)	J^π
15.6	ground	0.00	$7/2^-$
13.7	1st excited	1.94	$3/2^-$
13.3	2nd excited	2.46	$3/2^-$
13.1	3rd excited	2.58	$1/2^-$

Table 1. Predicted proton energies, ^{40}Ca(d, p)^{41}Ca at T_d = 10 MeV, θ_L = 30°. Levels from NNDC.

7. Threshold (Q < 0)

When $Q < 0$ the reaction has to pay for the extra rest mass, so there is a minimum beam energy E_{th} below which it simply cannot proceed. Right at threshold the products are created at rest in the CM, which means all of the available energy is used up: $T_{CM} = |Q|$. Feeding that into (3), $T_a\, m_A/(m_a + m_A) = |Q|$, and solving for the beam energy gives

$$E_{th} = |Q|\,(1 + m_a/m_A) \qquad (9)$$

The penalty depends on the mass ratio: for a heavy target ($m_A \gg m_a$) it is small, $E_{th} \approx |Q|$, while for equal masses it doubles to $2|Q|$. Just above threshold the products come out in a tight forward cone, which is what makes ^{7}Li(p, n)^{7}Be useful as a monoenergetic neutron source.

As a worked case, ^{7}Li(p, n)^{7}Be has a mass defect $\Delta m = -0.001766$ u, hence $Q \approx -1.644$ MeV. With $m_a/m_A = 1.008/7.016 = 0.144$, equation (9) gives $E_{th} = 1.644 \times 1.144 \approx 1.88$ MeV, against a measured 1.881 MeV (Marion & Young 1968): agreement to about 0.1%.

8. Is non-relativistic OK?

The whole derivation leaned on $p^2 = 2mT$, which is good to about 1% as long as $T < 0.1\, mc^2$. Past that you need the relativistic dispersion $E^2 = p^2c^2 + m^2c^4$ and are better off working with the invariant $s = (E_a + E_A)^2 - (p_a + p_A)^2c^2$. For the projectiles in this course the limits are:

Projectile	mc^2 (MeV)	Non-rel. good up to
proton	938	$T \approx 90$ MeV, covering most (p, p) and (p, n) work here
deuteron	1876	$T \approx 190$ MeV, so the 10 MeV example is well inside
alpha	3727	$T \approx 370$ MeV, so (α, α′) and (α, x) are safe
electron	0.511	always relativistic at these energies

In short, everything in Lectures 2–8 is comfortably non-relativistic; relativity only bites for electron scattering and above roughly T = 100 MeV/nucleon.

9. Inverse kinematics

Everything so far assumed a light beam on a heavy target. Modern radioactive-ion-beam facilities (FRIB, RIKEN-RIBF, GANIL-SPIRAL2, FAIR, IMP, ISOLDE) turn this around: a heavy, short-lived nucleus is accelerated onto a light target, usually a CD_2 foil or just hydrogen or helium. This is *inverse kinematics*, and the equations above all still hold; only the detection geometry changes.

Because $m_a \gg m_A$ now, the velocity v_{CM} approaches v_a, so everything is thrown forward in the LAB. In practice the light ejectile (say the proton from a (d, p) reaction on deuterium) turns up at *backward* laboratory angles even though it goes forward in the CM, while the heavy recoil flies straight ahead into a magnetic spectrometer that identifies it by mass and charge. This is the standard way to study nuclei far from stability, which cannot be made into a target in the first place, and it is behind the transfer, knock-out and spectroscopic-factor measurements of Lectures 6 to 8.

That is as far as kinematics takes us: it says where the products are allowed to go. How *often* a given channel actually happens is the cross section, which is where Lecture 3 begins.

Formula sheet

Formula	What it gives
(1) $v_{CM} = m_a v_a / (m_a + m_A)$	CM velocity (LAB ↔ CM)
(3) $T_{CM} = T_a m_A / (m_a + m_A)$	energy available in CM
(8) $Q = T_b(1 + m_b/m_B) - T_a(1 - m_a/m_B) - (2/m_B)\sqrt{(m_a m_b T_a T_b)} \cos \theta_L$	Q-value equation
(9) $E_{th} = \lvert Q\rvert(1 + m_a/m_A)$	threshold, $Q < 0$
$Q = (m_a + m_A - m_b - m_B) c^2$	Q from masses
1 u = 931.494 MeV/c^2	mass ↔ energy

References

- Satchler, *Introduction to Nuclear Reactions*, 2nd ed. (1990): §2.2, §2.4, §2.17, and the full derivation in Appendix B.
- Bertulani, *Nuclear Physics in a Nutshell* (2007): ch. 11.
- Krane, *Introductory Nuclear Physics* (1988): ch. 11.