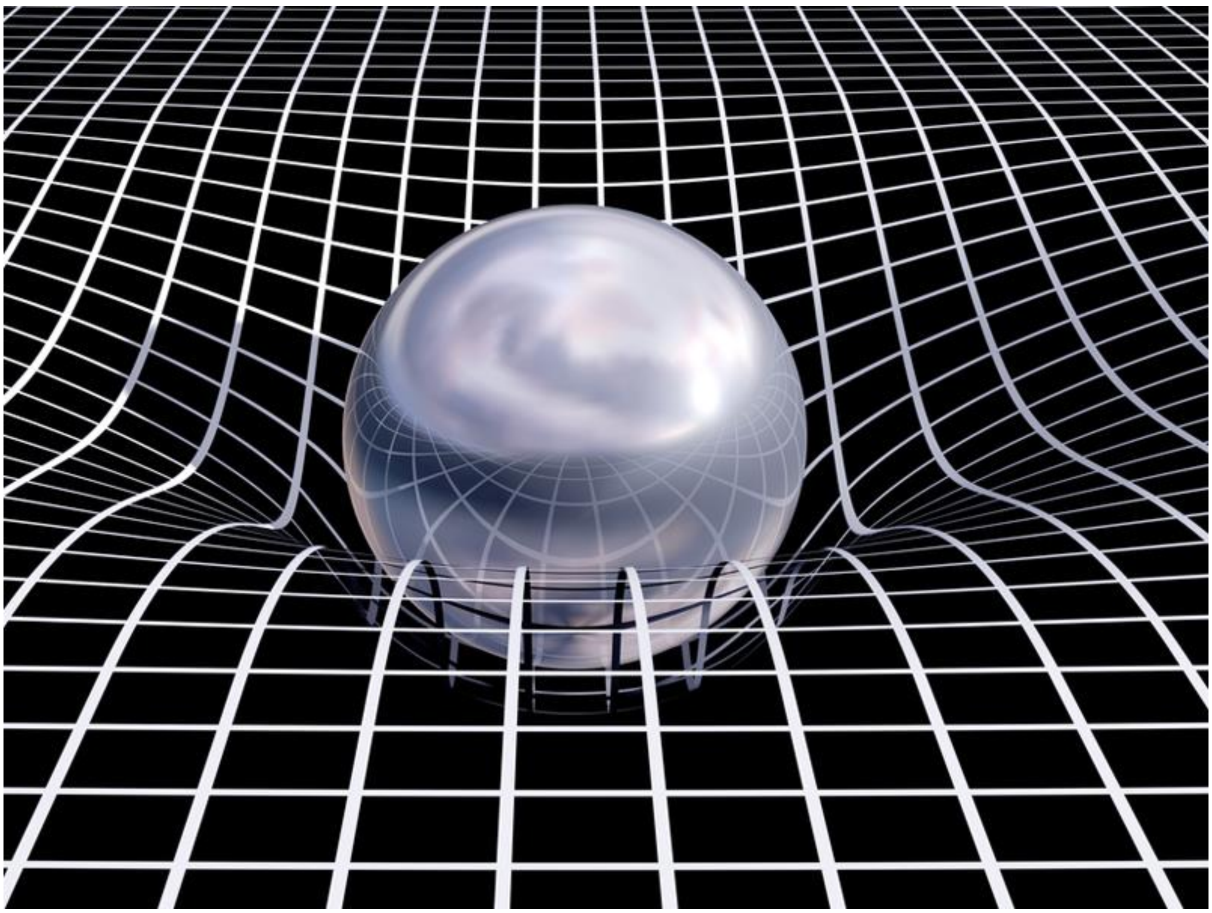
General Relativity



"Mass tells spacetime how to curve, Curved spacetime tells mass how to move"

General Relativity

General relativity

General relativity (**GR**, also known as the **general theory of relativity** or **GTR**) is the geometric theory of gravitation published by Albert Einstein in 1915 and the current description of gravitation in modern physics. General relativity generalizes special relativity and refines Newton's law of universal gravitation, providing a unified description of gravity as a geometric property of space and time, or spacetime. In particular, the *curvature of spacetime* is directly related to the energy and momentum of whatever matter and radiation are present. The relation is specified by the Einstein field equations, a system of partial differential equations.

Some predictions of general relativity differ significantly from those of classical physics, especially concerning the passage of time, the geometry of space, the motion of bodies in free fall, and the propagation of light. Examples of such differences include gravitational time dilation, gravitational lensing, the gravitational redshift of light, and the gravitational time delay. The predictions of general relativity in relation to classical physics have been confirmed in all observations and experiments to date. Although general relativity is not the only relativistic theory of gravity, it is the simplest theory that is consistent with experimental data. However, unanswered questions remain, the most fundamental being how *general relativity* can be reconciled with the laws of quantum physics to produce a complete and self-consistent theory of quantum gravity.

General theory of relativity based upon two postulates:

1) Principle of General Covariance

Equation expressing the laws of physics take the same form (covariant form) in all frame of reference regardless their state of motion.

The essential idea is that coordinates do not exist a priori in nature, but are only tricks used in describing nature, and hence should play no role in the formulation of fundamental physical laws.

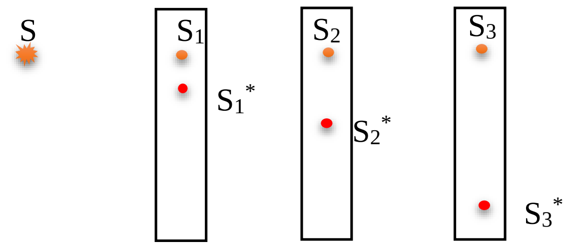
2) Principle of Equivalence

All physical phenomena that are observed within an accelerated system are identical with those occurring in a resting system, placed in a gravitational field, i.e. acceleration of a frame is equivalent to gravitation.

Thought Experiment:
Let a spaceship goes into a gravitation free region (no celestial body nearby).The rocket engine is shut off. The ship moves with constant velocity. Passengers, objects in the cabin floats freely.

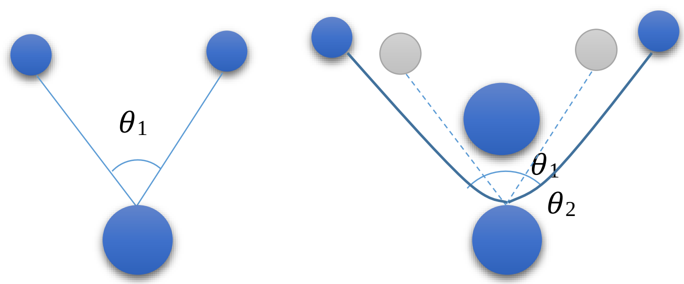
Let the engine starts again and spaceship get speed (accelerated). Things within cabin moves with constant velocity, but spaceship moves with greater velocity. So the things will be collected by the rear wall of the cabin and they will be pressed to it by force of acceleration. Travelers may rise up and make some experiment (dropping ball of different masses). They will get the same result and will feel that they are at rest on huge mass.

<u>So acceleration is equivalent to gravitation.</u>



A good theory explain the known facts and make predictions that can be checked by experiment.

We consider the passenger in the spaceship. They perform same optical experiment. Light goes along a straight path. If one directs a flash light to a wall the illuminated spot will be directly opposite. If this is done in an accelerated rocket the situation will be different. Let there are several transparent fluorescent flash rod at equal distances apart. When the rocket is in uniform motion light goes straight and make s_1, s_2, s_3 on the rod.



When the rocket is accelerated the spot will not be in a line. Light takes equal interval of time to go from one rod to another, but displacement of the rod will be in the ration 1:4:9.

The path will be parabola. As if a particle (say bullet) thrown in a gravitational field. Acceleration is equivalent to gravitation, so light will also bend in a gravitational field.

Predictions: Light should bend in a gravitational field, so photon should have mass.

Angular separation between two stars should differ when light comes through sun's gravitational field. In 1919 solar eclipses occurred 1.75'' deviation should have occurred but 1.77'' was observed.

Inertial and Gravitational Mass

There are several distinct phenomena which can be used to measure mass. Although some theorists have speculated that some of these phenomena could be independent of each other, current experiments have found no difference in results regardless of how it is measured:

- ***Inertial mass*** measures an object's resistance to being accelerated by a force (represented by the relationship $F = ma$).
- ***Active gravitational mass*** measures the gravitational force exerted by an object.
- ***Passive gravitational mass*** measures the gravitational force exerted on an object in a known gravitational field.

The mass of an object determines its acceleration in the presence of an applied force. The inertia and the inertial mass describe the same properties of physical bodies at the qualitative and quantitative level respectively, by other words, the mass quantitatively describes the inertia. According to Newton's second law of motion, if a body of fixed mass *m* is subjected to a single force *F*, then

$$F = m_i a$$

A body's mass also determines the degree to which it generates or is affected by a gravitational field. If a first body of mass m_g is placed at a distance *r* (center of mass to center of mass) from a second body of mass *M*, each body is subject to an attractive force

$$F_g = G\frac{Mm_g}{r^2}$$

This is sometimes referred to as gravitational mass.

Acceleration due to gravity

$$g = \frac{F_g}{m_i} = \frac{G\frac{Mm_g}{r^2}}{m_i} = \frac{GM}{r^2}\frac{m_g}{m_i}$$

This says that the ratio of gravitational to inertial mass of any object is equal to some constant *K* if and only if all objects fall at the same rate in a given gravitational field. This phenomenon is referred to as the "universality of free-fall". In addition, the constant *K* can be taken as 1 by defining our units appropriately.

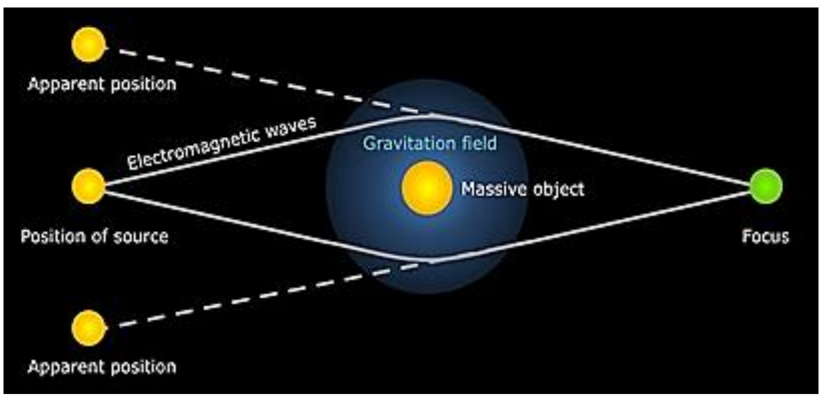
- ***Energy** also has mass according to the principle of mass–energy equivalence. This equivalence is demonstrated in a large number of physical processes including pair production, nuclear fusion, and the gravitational bending of light. Pair production and nuclear fusion are processes in which measurable amounts of mass are converted to energy, or vice versa. In the gravitational bending of light, photons of pure energy are shown to exhibit a behavior similar to passive gravitational mass.*

- ***Curvature of spacetime** is a relativistic manifestation of the existence of mass. Such curvature is extremely weak and difficult to measure. For this reason, curvature was not discovered until after it was predicted by Einstein's theory of general relativity. Extremely precise atomic clocks on the surface of the Earth, for example, are found to measure less time (run slower) when compared to similar clocks in space. This difference in elapsed time is a form of curvature called* *gravitational time dilation. Other forms of curvature have been measured using the Gravity Probe B satellite.*

Bending of Light in Gravitational Field

Newton's law of gravity predicts that gravity will not deflect light, which is massless; however, the principle of equivalence, on which general relativity is founded, predicts that light rays will be bent by gravity.

The following image shows the deflection of light rays that pass close to a spherical mass. To make the effect visible, this mass was chosen to have the same value as the Sun's but to have a diameter five thousand times smaller than the Sun's. According to general relativity, a light ray arriving from the left would be bent inwards, when viewed from the right, would differ by an angle (α, the deflection angle) whose size is inversely proportional to the distance (d) of the closest approach of the ray path to the center of mass.



Different theories of gravity give the same result for the angle of deflection

$$\alpha = \frac{GM}{dC^2} \approx 2.1 \times 10^{-6}\, radians$$

the only variation is in the value for the missing dimensionless constant.

Here are those values from exact calculation:

$$\alpha = \frac{GM}{dC^2} \times \begin{cases} 1 & (simplest\ guess) \\ 2 & (Newtonian\ gravity) \\ 4 & (Einstein\ theory) \end{cases}$$

where ‘1’ for the simplest guess and ‘2’ for Newtonian gravity is from integrating angular factors like cosine and sine that determine the position of the photon as it moves toward and past the sun. The most interesting constant is the ‘4’ for general relativity, which is twice the Newtonian value because light moves at the speed of light.

The extra bending is a consequence of Einstein's theory that formulates gravity in terms of the curvature of space-time. Newton's theory considers only time curvature but general relativity itself also calculates the space curvature. Since most objects move much slower than the speed of light, meaning that they travel much farther in time than in space, they feel mostly the time curvature. The Newtonian analysis is fine for those objects. Since light moves at the speed of light, it sees equal amounts of space and time curvature, so it bends twice as far as the Newtonian theory would predict.

Gravitational Red Shift

Consider a pulse of radiation (a photon) emitted by an atom A at rest in frame S (a space ship at rest on the earth's surface, e.g.). A uniform gravitational field g is directed downward in S, the photon falling down a distance d through this field before it is absorbed by the detector D (Fig.1a). To analyze what effect gravity has on the photon, let us consider the equivalent situation, shown in Fig. lb. Here we have an atom and a detector separated by a distance din a frame S' in which there is no gravitational field, the frame S' (a spaceship in outer space, e.g.) accelerating uniformly upward relative to S with a = g. When the photon is emitted, the atom has some speed u in this frame. The speed of the detector, when the photon reaches it, is u + at, where t is the time of flight of the photon. But t = d/c (approximately) and a = g so that the detector's speed on absorption is u + g(d/c). In effect, the detector has an approach velocity relative to the emitter of v = g d/c, independent of u. Hence, the frequency received, v', is greater than that emitted, v, the Doppler formula giving us

$$\frac{f'}{f} = \sqrt{\frac{c+v}{c-v}} = \sqrt{\frac{c+g\frac{d}{c}}{c-g\frac{d}{c}}} = \sqrt{\frac{\left(1+g\frac{d}{c^2}\right)^2}{1-\left(g\frac{d}{c^2}\right)^2}} \cong \sqrt{\frac{\left(1+g\frac{d}{c^2}\right)^2}{1}} = 1+\frac{gd}{c^2} \qquad (1)$$

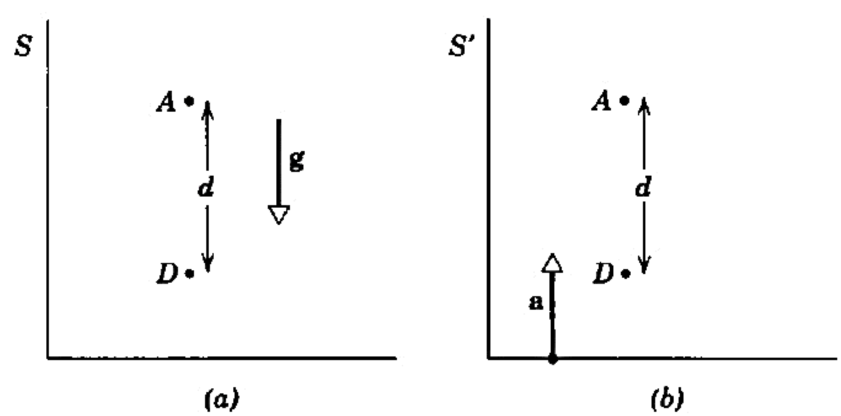


Fig. 1

By the principle of equivalence, we should obtain this same result in frame S. In this frame, however, A and D are at rest and there is no Doppler effect to explain the increase in frequency. There is a gravitational field, however, and the result in S' suggests that this field might act on the photon. Let us explore this possibility by ascribing to the photon a gravitational mass equal to its inertial mass, $m_i = \frac{E}{C^2}$. Then, in falling a distance d in a gravitational field of strength g, the photon gains energy $\frac{E}{C^2} gd$. How can we connect the energy E to the frequency v? In the quantum theory, the connection is $E = hf$. For the moment, let us use this relation so that the energy of the photon on absorption at D is its initial emission energy plus the energy gained in falling from A to D, or by $= hf + \frac{hf}{C^2} gd$. If we call this absorption energy $E' = hf'$, then we have

(2)

$$hf' = hf + \frac{hf}{C^2} gd$$

$$\frac{f'}{f} = 1 + \frac{gd}{C^2}$$

the same result obtained in frame S' (Eq. -1). Actually, it is not necessary to use quantum theory. We can show in relativity itself that E is proportional to v, because it follows, from the relativistic transformation of energy and momentum, that the energy in an electromagnetic pulse changes by the same factor as its frequency when observed in a different reference frame. **The conclusion then is that, in falling through a gravitational field, light gains energy and frequency** (its wavelength decreases and we say it is shifted toward the blue).

Clearly, had we reversed emitter and detector, we would have concluded that in rising against a gravitational field light loses energy and frequency (its wavelength increases and we say it is shifted toward the red). The predicted fractional change in frequency, *(f'— f)/f* or Δ*f/f*, is

$$\frac{f'}{f} - 1 = 1 + \frac{gd}{C^2} - 1$$

$$\frac{\Delta f}{f} = \frac{gd}{C^2}$$

even with d being the distance from sea level to the top of the highest mountains on earth, its value is only about 10^{-12}. Nevertheless, Pound and Rebka [2] in 1960 were able to confirm the prediction using the 74 ft high Jefferson tower at Harvard! For such a small distance we have

$$\frac{\Delta f}{f} = \frac{gd}{C^2} = \frac{9.8\, m/sec^2 \times 22.5\, m}{(3 \times 10^8)^2\, m/sec} \cong 2.5 \times 10^{-15}$$

an incredibly small effect. By using the Mossbauer effect (which permits a highly sensitive measurement of frequency shifts) with a gamma ray source, and taking admirable care to control the competing variables, Pound and Rebka observed this gravitational effect on photons and confirmed the quantitative prediction.

We can easily generalize our result (Eqs. 1 and 2) to photons emitted from the surface of stars and observed on earth. Here we assume that the gravitational field need not be uniform and that the result depends only on the difference in gravitational potential between the source and the observer. Then, in place of gd we have $\frac{GM}{R}$, where M is the mass of the star of radius R, and because the photon loses energy in rising through the gravitational field of the star, we obtain

$$\frac{f'}{f} = 1 + \frac{\frac{GM}{R}}{C^2}$$

$$f' = f\left(1 \pm \frac{GM}{RC^2}\right)$$

This effect is known as the gravitational red shift, for light in the visible pan of the spectrum will be shifted in frequency toward the red end. This effect is distinct from the Doppler red shift from receding stars. Indeed, because the Doppler shift is much larger, the gravitational red shift has not been confirmed with certainty.

Mach's Principle

The underlying idea in Mach's principle is that the origin of inertia or mass of a particle is a dynamical quantity determined by the environment, in particular the rest of the matter in the universe. This Principle actually implies that not only gravity but all physics should be formulated without reference to preferred inertial frames.

Mach's principle states that, the universe, as represented by the average motion of distant galaxies, does not appear to rotate relative to local inertial frames.

The idea is that the existence of absolute rotation (the distinction of local inertial frames vs. rotating reference frames) is determined by the large-scale distribution of matter, as illustrated by this story:

You are standing in a field looking at the stars. Your arms are resting freely at your side, and you see that the distant stars are not moving. Now start spinning. The stars are whirling around you and your arms are pulled away from your body. Why should your arms be pulled away when the stars are whirling? Why should they be dangling freely when the stars don't move?

Mach's principle says that this is not a coincidence—that there is a physical law that relates the motion of the distant stars to the local inertial frame. If you see all the stars whirling around you, Mach suggests that there is some physical law which would make it so you would feel a centrifugal force. There are a number of rival formulations of the principle. It is often stated in vague ways, like "mass out there influences inertia here". A very general statement of Mach's principle is "local physical laws are determined by the large-scale structure of the universe".

This concept was a guiding factor in Einstein's development of the general theory of relativity. Einstein realized that the overall distribution of matter would determine the metric tensor, which tells you which frame is rotationally stationary.

Correspondence Principle

The term "correspondence principle" is used in a more general sense to mean the reduction of a new scientific theory to an earlier scientific theory in appropriate circumstances. This requires that the new theory explain all the phenomena under circumstances for which the preceding theory was known to be valid, the "correspondence limit".

For example,

- In the case of Einstein's special relativity correspondence principle states that, special relativity reduces to classical mechanics in the limit of velocities small compared to the speed of light ;
- In the case of General relativity correspondence principle states that, General relativity reduces to Newtonian gravity in the limit of weak gravitational fields;

In order for there to be a correspondence, the earlier theory has to have a domain of validity—it must work under *some* conditions. Not all theories have a domain of validity. For example, there is no limit where Newton's mechanics reduces to Aristotle's mechanics because Aristotle's mechanics, although academically dominant for 18 centuries, does not have any domain of validity.

Example: Relativistic kinetic energy

Here we will show that the expression of kinetic energy from special relativity becomes arbitrarily close to the classical expression, for speeds that are much slower than the speed of light, $v \ll c$.

Einstein's mass-energy equation

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Where the velocity, v is the velocity of the body relative to the observer, m_0 is the *rest* mass (the observed mass of the body at zero velocity relative to the observer), and c is the speed of light.

When the velocity v vanishes or extremely low compared to C, the energy expressed above is not zero, and represents the *rest* energy,

$$E_0 = m_0 c^2$$

When the body **is** in motion relative to the observer, the total energy exceeds the rest energy by an amount that is, by definition, the *kinetic* energy,

$$T = E - E_0 = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0c^2$$

Using the approximation

$$(1+x)^n \approx 1 + nx \quad \text{When } |x| \ll 1$$

We get, when speeds are much slower than that of light, or $v \ll c$,

$$T = m_0c^2\left\{\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right\}$$

$$= m_0c^2\left\{1 - \left(-\frac{1}{2}\right)\frac{v^2}{c^2} - 1\right\}$$

$$= \frac{1}{2}m_0v^2$$

Which is the Newtonian expression for kinetic energy.

Geodesic Equation

In general relativity, a **geodesic** generalizes the notion of a "straight line" to curved spacetime. Importantly, the world line of a particle moving under gravitation and free from all external, non-gravitational force is a particular type of geodesic. In other words, a freely moving or falling particle always moves along a geodesic.

In general relativity, gravity can be regarded as not a force but a consequence of a curved spacetime geometry where the source of curvature is the stress–energy tensor (representing matter, for instance). Thus, for example, the path of a planet orbiting a star is the projection of a geodesic of the curved 4-D spacetime geometry around the star onto 3-D space.

The full **geodesic equation** is this:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 .$$

where s is a scalar parameter of motion (e.g. the proper time), and $\Gamma^{\mu}_{\alpha\beta}$ are Christoffel symbols (sometimes called the affine connection coefficients or Levi-Civita connection coefficients) which is symmetric in the two lower indices. Greek indices may take the values: 0, 1, 2, 3 and the summation convention is used for repeated indices α and β.
The quantity on the left-hand-side of this equation is the acceleration of a particle, and so this equation is analogous to Newton's laws of motion which likewise provide formulae for the acceleration of a particle. This equation of motion employs the Einstein notation, meaning that repeated indices are summed (i.e. from zero to three). The Christoffel symbols are functions of the four space-time coordinates, and so are independent of the velocity or acceleration or other characteristics of a test particle whose motion is described by the geodesic equation.

Einstein notation

According to this convention, when an index variable appears twice in a single term and is not otherwise defined (see free and bound variables), it implies summation of that term over all the values of the index. So where the indices can range over the set {1, 2, 3},

$$y = \sum_{i=1}^{3} c_i x^i = c_1 x^1 + c_2 x^2 + c_3 x^3$$

is simplified by the convention to:

$$y = c_i x^i$$

The upper indices are not exponents but are indices of coordinates, coefficients or basis vectors. That is, in this context x^2 should be understood as the second component of **x** rather than the square of **x** (this can occasionally lead to ambiguity). The upper index position in x^i is because, typically, an index occurs once in an upper (superscript) and once in a lower (subscript) position in a term (see 'Application' below). And typically (x^1, x^2, x^3) would be equivalent to the traditional (x, y, z).

In general relativity, a common convention is that

- the Greek alphabet is used for space and time components, where indices take on values 0, 1, 2, or 3 (frequently used letters are μ, ν, ...),
- the Latin alphabet is used for spatial components only, where indices take on values 1, 2, or 3 (frequently used letters are i, j, ...),

Einstein Field Equation

1. Gravitation and acceleration are equivalence (principle of equivalence)

2. Light will to be appear to bends in gravitational field

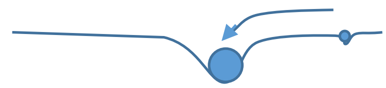


Fig. 1

From Newton's law of gravitation, the gravitational force $F = \frac{GMm}{r^2}$. In the case of light, which is bending in a gravitational field, sun has mass but photon has no mass. So we cannot use this formula for bending of light. Einstein came up with completely different approach of gravity. He says actually what is happening here is, all form of motion are represented by the movement in the curved spacetime.
Let us consider a trampoline which representing spacetime. We put a marble and a big mass on the trampoline then, the marble affect very tiny indentation. For the big mass, it will create large indentation and the marble moves towards big mass. Now newton said, this movement towards big mass is caused by gravitational attraction. On the other hand Einstein says marble is moving along the shortest path in the curved spacetime.

The Einstein field equation is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = \frac{8\pi G}{C^4} T_{\mu\nu}$$

μ *and* ν represents the dimension of the spacetime 0, 1, 2 and 3. (0= time, 1=x, 2=y, 3=z axis)

$R_{\mu\nu} =$ Ricci Curvature Tensor

$g_{\mu\nu}$ = Metric Tensor

R = Curvature Scalar

Λ= Cosmological Constant

$T_{\mu\nu}$ = Stress Energy Momentum Tensor

G = Gravitational constant

C = Speed of Light

Everything in the left hand side refers to the curvature of the spacetime and everything in the right hand side refers to the mass and energy. What Einstein field equations are basically say is, "Mass tells spacetime how to curve, Curved spacetime tells mass how to move".

1. Introduction
2. Metric tensor
3. Cristoffel symbol
4. Curvature- Ricci tensor, Curvature scalar

5. Stress energy momentum tensor
6. Cosmological constant
7. Put it all together

Metric Tensor

Physical laws must be independent of any particular coordinate systems used in describing them mathematically, if they are to be valid. A study of the consequences of this requirement leads to tensor analysis, of great use in general relativity theory, differential geometry, mechanics, elasticity, hydrodynamics, electromagnetic theory and numerous other fields of science and engineering.

Let us consider a field may be magnetic, electric, temperature field. Here we consider one of the most basic field, a field with grass, which is very bumpy field and the value of the field is the height above from the sea level denoted by ϕ.

In order to locate a position in this bumpy field, we imposed a set of coordinates x and y. Now question is how this does this height changes as the point moves a little bit. This depends on the direction of movement because the field is bumpy and very irregular. Let us consider a 1:10 gradient. i.e we will drop 1 meter down in 10 meter.

$d\phi_s = \frac{d\phi}{dx}dx$ $\qquad\qquad$ $d\phi_s = \frac{1}{10}5 = \frac{1}{2}$

But in our case, the field is very irregular, so for very small movement we can express the change in height in the x direction as

$$d\phi_x = \frac{d\phi}{dx}dx$$

Similarly for movement in the y direction

$$d\phi_y = \frac{d\phi}{dy}dy$$

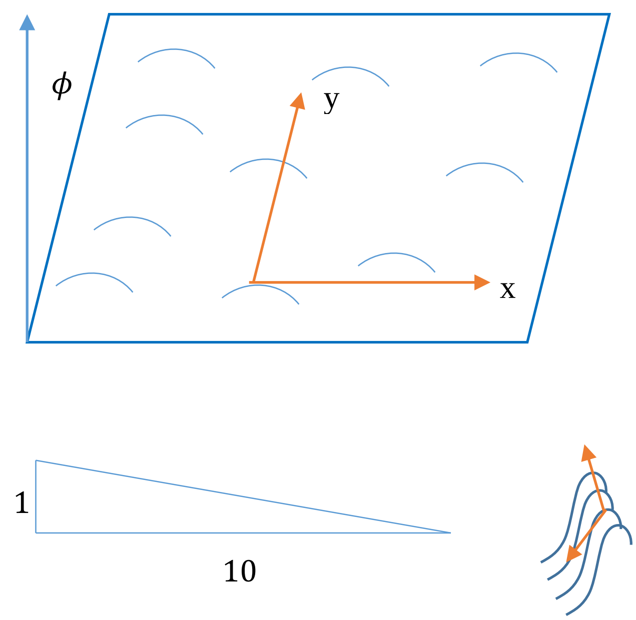


Fig. 2

The gradients in these two equations are not same. Now if we walk in arbitrary direction combination of x and y, instead of one direction then the height then the change in height

$$d\boldsymbol{\phi_s} = d\boldsymbol{\phi_x} + d\boldsymbol{\phi_y}$$

$$d\boldsymbol{\phi_s} = \frac{d\phi}{dx}d\boldsymbol{x} + \frac{d\phi}{dy}d\boldsymbol{y}$$

For proper writing,

$$d\boldsymbol{\phi_s} = \frac{\partial \phi}{\partial x} d\boldsymbol{x} + \frac{\partial \phi}{\partial y} d\boldsymbol{y}$$

As the ϕ may depends on another variables.

Up to now we have used x and y. Now we change

$$x \to x^1$$

$$y \to x^2$$

Why do we do that? For thre dimensional coordinate we x,y,z and for more dimension we use more alphabet which may use differently in physics. This will creates confusion. Through this numbering we can extend unlimited number of coordinate and use the alphabets in another way. Now we rewrite the above equation in terms of our new coordinates as

$$d\boldsymbol{\phi_s} = \frac{\partial \phi}{\partial x^1} d\boldsymbol{x^1} + \frac{\partial \phi}{\partial x^2} d\boldsymbol{x^2} + \cdots \ldots \ldots \ldots$$

$$d\boldsymbol{\phi_s} = \sum_{\boldsymbol{n}} \frac{\partial \phi}{\partial x^n} d\boldsymbol{x^n} \qquad (1)$$

(For n-dimensional coordinate system)

In the case of relativity, either special or general, every physical laws must be independent of reference. We know different observers measures different distances and times in their reference depending on the relative velocity, which are called length contraction and time dilation.

If you something have to universally true, then it must be true in all reference

Let we set a $x^1 - x^2$ coordinate system and the position p defined by (x^1, x^2). Another one can set $y^1 - y^2$ coordinate system. Even p point in the same place, x^1, x^2 and y^1, y^2 coordinates are different. If we want to express y^1 in terms of x coordinates we have to express in terms of x^1, x^2. i.e.

$$y^1 = y^1(x^1, x^2)$$

$$y^2 = y^2(x^1, x^2)$$

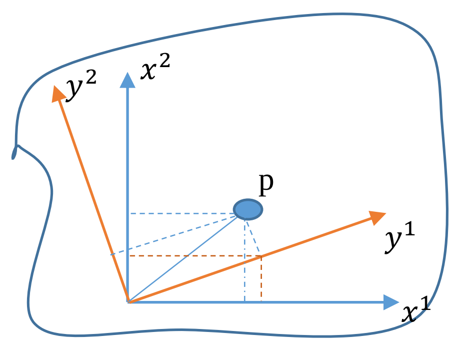


Fig. 3

If we know all the gradients in the x frame, how can we find the all the gradients in the y frame and how can we relate the two?

For this we use the chain rule

$$\frac{d\phi}{dy^1} = \frac{\partial \phi}{\partial x^1}\frac{dx^1}{dy^1} + \frac{\partial \phi}{\partial x^2}\frac{dx^2}{dy^1} + \cdots \ldots \ldots \ldots$$

So we see that, to find the y^1 we have to know all the x coordinates

We can summarizes as

(2)

$$\frac{d\phi}{dy^n} = \sum_m \frac{\partial\phi}{\partial x^m}\frac{dx^m}{dy^n}$$

A **tensor** is an algebraic object related to a vector space and its dual space that can be defined in several different ways, often a scalar, tangent vector at a point, a cotangent vector (dual vector) at a point or a multi-linear map from vector spaces to a resulting vector space. Euclidean vectors and scalars are the simplest tensors. While tensors are defined independent of any basis, the literature on physics often refers to them by their components in a basis related to a particular coordinate system.

Any vector space V has a corresponding ***dual vector space*** *(or just* ***dual space*** *for short) consisting of all linear functionals on V, together with the vector space structure of point wise addition and scalar multiplication by constants.*

Scalar - magnitude – Tensor of rank 0

Vector – Magnitude - Tensor of rank 1

How does a vector transform? We can use eqn. 1 with slight adjustment to it.
Let a n dimension vector V in the y frame related with the m dimension vector V in the x frame as

$$V_y^n = \sum_m \frac{\partial y^n}{\partial x^m} V_x^m \tag{3}$$

Where m is a dummy variable.

The tensor that will be interested in is tensor of rank 2, combination of two vector having fixed relationship. Let us take an example, the work done $w = \boldsymbol{F}.\boldsymbol{x} = F cos\theta\ x = 0\ (\,for\ \theta = 90°)$. So we have got essentially something which combines two vectors. Here is the thing, if the value of work done (value of a tensor) is zero in one frame of reference, it will zero in all frame of reference, i.e. if the block doesn't move in one frame, it doesn't move in any frame. Tensor is a fixed relationship between two vectors. That is why tensors are so important in Einstein's field equations.

Let us consider combination of m dimensional vector A^m and n dimensional vector B^n. Combination of this vector is a tensor T^{mn}.

$$T^{mn} = A^m B^n$$

In two dimensional space m and n can be either can have value 1 or 2, then T will have 4 versions. For 3 dimensional space, T^{mn} has 9 versions. Using eqn. 3, we can write,

$$A_y^m B_y^n = \sum_r \frac{\partial y^m}{\partial x^r} A_x^r \sum_s \frac{\partial y^n}{\partial x^s} B_x^s$$

$$T_y^{mn} = \sum_{rs} \frac{\partial y^m}{\partial x^r}\frac{\partial y^n}{\partial x^s} A_x^r B_x^s$$

(4)

$$T_y^{mn} = \sum_{rs} \frac{\partial y^m}{\partial x^r} \frac{\partial y^n}{\partial x^s} T_x^{rs}$$

This transformation from y to x frame is called contravariant transformation. There is another transformation, covariant transformation which is,

$$T_{mn}^y = \sum_{rs} \frac{\partial x^r}{\partial y^m} \frac{\partial x^s}{\partial y^n} T_{rs}^x \tag{5}$$

According to Pythagoras,

$$ds^2 = {dx^1}^2 + {dx^2}^2 \ldots\ldots$$

$$= \sum_m dx^m dx^m$$

Again, $ds^2 = \sum_{mn} dx^m dx^n$ will not be right. Then in proper way we can write,

$$ds^2 = \sum_{mn} dx^m dx^n \, \delta_{mn} \tag{5a}$$

Where δ_{mn} is the ***Kronecker delta***, is a function of two variables, usually just non-negative integers. The function is 1 if the variables are equal, and 0 otherwise.

Now rewriting the equation 1 in slightly different form as

$$dx^m = \sum_r \frac{\partial x^m}{\partial y^r} dy^r$$

$$dx^m = \frac{\partial x^m}{\partial y^r} dy^r$$

Similarly, $dx^n = \frac{\partial x^m}{\partial y^s} dy^s$

Now substituting this value in eqn. 5a,

$$ds^2 = \delta_{mn} \sum_{mn} \frac{\partial x^m}{\partial y^r} dy^r \frac{\partial x^n}{\partial y^s} dy^s$$

$$= \delta_{mn} \sum_{mn} \frac{\partial x^m}{\partial y^r} \frac{\partial x^n}{\partial y^s} dy^r dy^s$$

$$= g_{mn} dy^r dy^s$$

Where $g_{mn} = \delta_{mn} \sum_{mn} \frac{\partial x^m}{\partial y^r} \frac{\partial x^m}{\partial y^s}$ is called the metric tensor. a metric tensor is a type of function which takes as input a pair of tangent vectors **v** and **w** at a point of a surface (or higher dimensional differentiable manifold) and produces a real number scalar g(**v, w**) in a way that generalizes many of the familiar properties of the dot product of vectors in Euclidean space. In the same way as a dot product, metric tensors are used to define the length of and angle between tangent vectors.

In flat space g_{mn} simply reduces to Kronecker delta δ_{mn} term. Pythagoras theorem is true in flat space for right angled triangle, but if you draw a triangle on the surface of a sphere, then it need correction. You may took metric tensor as a correction to Pythagoras theorem for curved spacetime.

In the Einstein's field equation $g_{\mu\nu}$ terms refers the metric tensor which includes spacetime.

Cristoffel Symbol:

Remember the important thing about tensors is the fixed relationship between two vectors and they are independent of coordinate systems. Let us suppose a tensor w having n m coordinates in x frame is equal to another tensor v having n m coordinates in x frame, i.e.,

$$W_{nm}(x) = V_{nm}(x)$$

then this is true for all frame of reference. But the problem is the derivatives of tensor does not transform necessarily between frames.

Let's a tensor T_{mn} in the x frame.

$$T_{mn}(x) = \frac{\partial V_m(x)}{\partial x^n}$$

Now the question is, does the same tensor in the y frame

$$T_{mn}(y) = \frac{\partial V_m(y)}{\partial y^n}$$

is true? Answer is no.

Dropping the summation term as Einstein did, equation 5 can be written as,

$$T_{mn}^{y} = \frac{\partial x^r}{\partial y^m}\frac{\partial x^s}{\partial y^n}T_{rs}^{x}$$

$$= \frac{\partial x^r}{\partial y^m}\frac{\partial x^s}{\partial y^n}\frac{\partial V_r(x)}{\partial x^s}$$

$$T_{mn}^{y} = \frac{\partial x^r}{\partial y^m}\frac{\partial V_r(x)}{\partial y^n}$$

So we see that it is not equal to $T_{mn}(y)$. Now, again using equation 3

$$\frac{\partial V_r(x)}{\partial y^n} = \frac{\partial}{\partial y^n}\left(\frac{\partial x^r}{\partial y^m}V_r(x)\right)$$

(6)

$$= \frac{\partial x^r}{\partial y^m}\frac{\partial V_r(x)}{\partial y^n} + \frac{1}{\partial y^n}\frac{\partial x^r}{\partial y^m}V_r(x)$$

Now the combination of differentials in the second term is called gamma Γ^r_{nm} and this symbol is called the Cristoffel symbol. This is simply a shorthand way of writing this double differential.

So ,

$$T_{mn}(y) \neq \frac{\partial V_m(x)}{\partial y^n}$$

because of the extra term, the gamma. Because the tensor $T_{mn}(y)$ equal to covariant derivative of vector V_m. What is covariant derivative? Covariant derivative is equal to the ordinary derivative plus the correction term (Crystoffel symbol) multiplied by $V_r(x)$

$$T_{mn}(y) = \nabla_n V_m = \frac{\partial V_m}{\partial y^n} + \Gamma^r_{nm}V_r(x) \tag{7}$$

So we now got transformation equation of derivatives which has Crystoffel symbol as an extra term.

Let us take the covariant derivative of the tensor T_{mn} for two indices from eqn. 7

$$\nabla_p T_{mn} = \frac{\partial T_{mn}}{\partial y^p} + \Gamma^r_{pm}T_{rn} + \Gamma^r_{pn}T_{mr} \tag{8}$$

This is the transformation of tensor from one frame to another.

Now let us take the covariant derivative of metric tensor we derived earlier

$$\nabla_r g_{mn}(x) = \nabla_p \delta_{mn} \sum_{mn} \frac{\partial x^m}{\partial y^r}\frac{\partial x^m}{\partial y^s}(x)$$

As we know that the metric in flat space reduces to Kronecker delta, its value will be either 0 or 1.

So

$$\nabla_r g_{mn}(x) = 0$$

From the property of tensor we can say that, this covariant derivative of metric tensor will be zero in all frame of reference. Substituting T_{mn} by g_{mn} in equation 8

$$\nabla_p g_{mn} = \frac{\partial g_{mn}}{\partial y^p} + \Gamma^r_{pm}g_{rn} + \Gamma^r_{pn}g_{rm} = 0$$

Now we got an equation which has metric tensor and derivative of metric tensor and the Cristoffel symbol in it. Now let we rearrange this equation so that we can express the Cristoffel symbol in terms of metric tensor.

(9)

$$\Gamma^a_{bc}(x) = \frac{1}{2} g^{ad} \left\{ \frac{\partial g_{dc}}{\partial x^b} + \frac{\partial g_{ab}}{\partial x^c} - \frac{\partial g_{bc}}{\partial x^d} \right\}$$

Where d is an additional term for additional summation term. The Cristoffel symbol is not itself a tensor, it is a correction term. This will be equal to zero in flat space and not equal to zero in curvilinear space. It can be expressed wholly a metric tensor and the first derivatives of metric tensor.

You might asked yourself why we have spent all this time trying to find a thing with the Cristoffel symbol when it doesn't even fetch in the Einstein field equation? That's because the Cristoffel symbol is buried under the Ricci curvature tensor.

Curvature:

Let us take a path in flat space and take a vector. Then the vector is parallel transported around the path. That means you keep the same length of the vector and you keep it oriented in same way. So it travels parallel to itself. All of the vectors are parallel until you get back to you started.

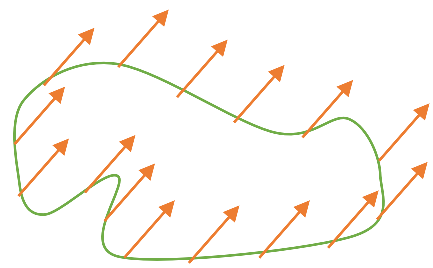


Fig. 4

Now let's take a cut on a cone as shown as a dotted line in the cone. Before cutting A and B are same point.

After opened it up, OA and OB are different lines. Now if we parallel transport a vector (along the line OB) starting from B then at the end it makes the different angle in opened cone of amount α. This gives the measure of curvature.

If you parallel transport of a vector around a surface, you get back to the same point but the vector has change direction, then this surface is curved.

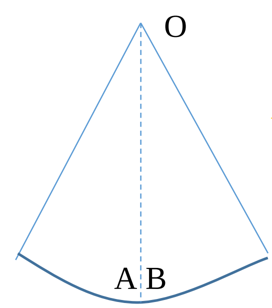


Fig. 5

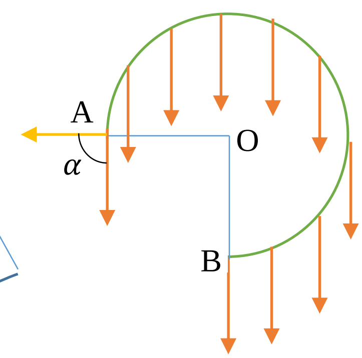


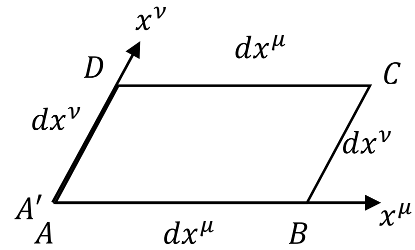
Fig. 6

Commutator:

[A, B] = AB –BA

$$\left[\frac{\partial}{\partial x}, f(x)\right] = \frac{\partial}{\partial x} f(x) - f(x)\frac{\partial}{\partial x} = 1\frac{\partial f(x)}{\partial x} + f(x)\frac{\partial 1}{\partial x} - f(x)\frac{\partial}{\partial x} = \frac{\partial f(x)}{\partial x}$$

Now let's take two coordinates x^{μ}, x^{ν} that's not have to be right angled.



In this x^{μ}, x^{ν} coordinate system ABCD is a parallelogram of sides dx^{μ} and dx^{ν}. Let us take parallel transform from A to B to C to A'. Because we don't know whether point A and A' are same or not. So the vector at A may not equal to vector at A' for curved surface.

Now let us compare two paths

$$(V_C - V_D) - (V_B - V_A) = 0 \qquad \text{[For flat surfaces]}$$

$$(V_C - V_D) - (V_B - V_A) = Difference\ in\ x^{\mu}\ direction \qquad \text{[For curved surfaces]}$$

Similarly

$$(V_C - V_B) - (V_D - V_{A'}) = Difference\ in\ x^{\nu}\ direction$$

Now let us take the difference of differences

$$(V_C - V_D) - (V_B - V_A) - \{(V_C - V_B) - (V_D - V_{A'})\} = dV$$

$$V_A - V_{A'} = dV$$

We can write $V_C - V_D$ as

$$V_C - V_D = \frac{\partial V}{\partial x^{\mu}} dx^{\mu} V \qquad \text{[For flat surfaces]}$$

But for curved space we have to use covariant derivative to be more accurate, so

$$V_C - V_D = \frac{\partial V}{\partial x^{\mu}} dx^{\mu} V = \nabla_{\mu} dx^{\mu} V$$

So the differences in curved surface are

$$(V_C - V_D) - (V_B - V_A) = \nabla_{\nu} dx^{\nu} \nabla_{\mu} dx^{\mu} V$$

$$(V_C - V_B) - (V_D - V_{A'}) = \nabla_{\mu} dx^{\mu} \nabla_{\nu} dx^{\nu} V$$

So we can write

$$\begin{aligned} dV &= \nabla_{\nu} dx^{\nu} \nabla_{\mu} dx^{\mu} V - \nabla_{\mu} dx^{\mu} \nabla_{\nu} dx^{\nu} V \\ &= dx^{\nu} dx^{\mu} V \left(\nabla_{\nu}\nabla_{\mu} - \nabla_{\mu}\nabla_{\nu}\right) \\ &= dx^{\nu} dx^{\mu} V \left[\nabla_{\nu}, \nabla_{\mu}\right] \qquad (9a) \end{aligned}$$

Again we know, $\quad \nabla_{\nu} = \partial_{\nu} + \Gamma_{\nu} \qquad [\frac{\partial}{\partial x^{\nu}} = \partial_{\nu}]$

Then the commutator

$$\begin{aligned}
[\nabla_\nu, \nabla_\mu] &= (\partial_\nu + \Gamma_\nu)(\partial_\mu + \Gamma_\mu) - (\partial_\mu + \Gamma_\mu)(\partial_\nu + \Gamma_\nu) \\
&= \partial_\nu \partial_\mu + \partial_\nu \Gamma_\mu + \Gamma_\nu \partial_\mu + \Gamma_\nu \Gamma_\mu - (\partial_\mu \partial_\nu + \partial_\mu \Gamma_\nu + \Gamma_\mu \partial_\nu + \Gamma_\mu \Gamma_\nu) \\
&= (\partial_\nu \partial_\mu - \partial_\mu \partial_\nu) + (\partial_\nu \Gamma_\mu - \Gamma_\mu \partial_\nu) + (\Gamma_\nu \partial_\mu - \partial_\mu \Gamma_\nu) + (\Gamma_\nu \Gamma_\mu - \Gamma_\mu \Gamma_\nu) \\
&= 0 + [\partial_\nu, \Gamma_\mu] - [\partial_\mu, \Gamma_\nu] + [\Gamma_\nu, \Gamma_\mu] \\
&= \frac{\partial \Gamma_\mu}{\partial x^\nu} - \frac{\partial \Gamma_\nu}{\partial x^\mu} + [\Gamma_\nu, \Gamma_\mu]
\end{aligned}$$

These entire term is called the Riemann Tensor. This made up of Cristoffel symbol and derivatives of Cristoffel symbol. For our purposes, this can also be regarded as Ricci Tensor $\mathrm{R}_{\mu\nu}$, which is first term in the Einstein field equation.

So the change in vector is

$$dV = dx^\nu dx^\mu V \mathrm{R}_{\mu\nu}$$

We just show, Ricci tensor is made up of Cristoffel symbol and Cristoffel symbols are made up of metric tensor and derivative of metric tensor and metric tensors are device to need to correct Pythagoras in curved surface.

From the Ricci tensor $\mathrm{R}_{\mu\nu}$ we can derive a scalar R called curvature scalar. If the curvature scalar is not zero then the surface is not flat.

Stress Energy Momentum Tensor ($\mathbf{T}_{\boldsymbol{\mu\nu}}$)

The shortest distance between two points in the surface of a sphere is called geodesic. For example the equator is geodesic. We can define a tangent vector, the rate of change of distance w.r.to proper time as $\frac{dx^\mu}{d\tau}$. In order for this distance is to be shortest, this derivative should be zero i.e. $\frac{dx^\mu}{d\tau} = 0$.

As we know, we have to use covariant derivative for curved space, let us take the covariant derivative of this tangent vector

$$\nabla \frac{dx^\mu}{d\tau} = \frac{\partial}{\partial \tau} \frac{\partial x^\mu}{\partial \tau} + \Gamma = 0$$

$$\frac{\partial^2 x^\mu}{\partial \tau^2} = -\Gamma = acceleration$$

From Newton's law, we know $a = \frac{F}{m} = -\Gamma$

Now we know the Cristoffel symbol

$$\Gamma^a_{bc}(x) = \frac{1}{2} g^{ad} \left\{ \frac{\partial g_{dc}}{\partial x^b} + \frac{\partial g_{ab}}{\partial x^c} - \frac{\partial g_{bc}}{\partial x^d} \right\}$$

The general relativity and Newton's law of gravitation must coincide for ordinary masses, gravity and speed. When that happens, g^{ad} becomes 1 and the derivatives becomes very very small, near to zero except the time component $\frac{\partial g_{00}}{\partial x}$ of the metric tensor.

So above eqn becomes

$$\Gamma^a_{bc}(x) = \frac{1}{2}\frac{\partial g_{00}}{\partial x} \equiv F$$

For unit mass

But in Newtonian mechanics, force is negative gradient of potential, i.e.,

$$F = -\frac{d\varphi}{dx} \qquad \varphi = mgx$$

-ve sign indicates opposite direction of F and x

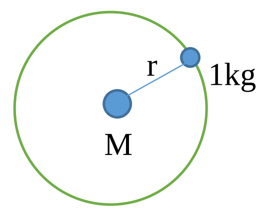
So the Cristoffel symbol

$$\Gamma^a_{bc}(x) = \frac{1}{2}\frac{\partial g_{00}}{\partial x} = \frac{d\varphi}{dx}$$

$$\frac{\partial g_{00}}{\partial x} = 2\frac{d\varphi}{dx}$$

$$g_{00} = 2\varphi + constant$$

For three dimension,



$$F = -\frac{GMm}{r^2} \qquad F = -\nabla\varphi$$

Force on unit mass

$$F = -\frac{GM}{r^2}$$

Now the force capability across the whole sphere is

$$\int F.dA = -\frac{GM}{r^2}\int dA = -\frac{GM}{r^2}4\pi r^2 = -4\pi GM \qquad (9b)$$

Now from divergence theorem we can write,

$$\int \boldsymbol{F}.d\boldsymbol{A} = \int \boldsymbol{\nabla}.\boldsymbol{F}dV$$

The density $\rho = \frac{M}{V}$ and the mass $M = \int \rho dV$

So from eqn. 9b

$$\int F.dA = -4\pi GM$$

$$\int \nabla . \boldsymbol{F} dV = -4\pi G \int \rho dV$$

$$\nabla . \boldsymbol{F} = -4\pi G\rho$$

$$-\nabla . \nabla\varphi = -4\pi G\rho$$

$$\nabla^2\varphi = 4\pi G\rho$$

Earlier we shown that $g_{00} = 2\varphi + constant$. Let we drop the constant, then,

$$g_{00} = 2\varphi$$

$$\varphi = \frac{1}{2} g_{00}$$

So we get,

$$\nabla^2 \frac{1}{2} g_{00} = 4\pi G\rho$$

$$\nabla^2 g_{00} = 8\pi G\rho \tag{9c}$$

Now the problem is, it is not a tensor equation, and for general relativity we need tensor equations. So what we looking for is something similar to it. So we want something that has the form $G_{\mu\nu}$, the Einstein's tensor, on the left side and $T_{\mu\nu}$, that contains all the mass, energy, pressure, stress terms, on the right side.

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{9d}$$

We know the momentum 4-vector can be expressed as

$$p = m\left(\frac{x_0}{\tau}, \frac{x_1}{\tau}, \frac{x_2}{\tau}, \frac{x_3}{\tau}\right)$$

$$mc^2, mv_x, mv_y, mv_z$$

This is essentially a vector but we need tensor $T_{\mu\nu}$

T_{00}	T_{01}	T_{02}	T_{03}
T_{10}	T_{11}	T_{12}	T_{13}
T_{20}	T_{21}	T_{22}	T_{23}
T_{30}	T_{31}	T_{32}	T_{33}

T_{00} is the time component of the tensor

T_{01}, T_{02}, T_{03} are called energy flow paths of the tensor

T_{10}, T_{20}, T_{30} are called momentum density of the tensor

And rest 9 components are essentially momentum, flux, stress, pressure parts of the tensor

Now see that, every part of the energy that we can think of, the pure energy, rest mass energy, momentum, stress all of it somehow find a place in this stress energy momentum tensor.

If we like to measure the energy per unit volume

$$\frac{E}{V} = \frac{F.L}{L^3} = \frac{F}{A} = p$$

Now the right hand side of the eqn. 9d is $8\pi G T_{\mu\nu}$ which is mass term, so left side must be curvature term. Einstein thought this can be Ricci curvature tensor.

$$R_{\mu\nu} = 8\pi G T_{\mu\nu}$$

But the problem is, energy neither be created nor be destroyed.

If we take the derivative $\partial T_{\mu\nu} = 0$

$$\partial R_{\mu\nu} \neq 0$$

We must always use the covariant derivative. So, if we got the covariant derivative of energy side is zero

$$\nabla(8\pi G T_{\mu\nu}) = 0 \tag{10}$$

$$\nabla T_{\mu\nu} = 0$$

we need something on the left side whose covariant derivative will also be zero. Einstein found that the covariant derivative of Ricci tensor is not zero but

$$\nabla \mathrm{R}_{\mu\nu} = \frac{1}{2}\nabla \mathrm{g}_{\mu\nu} R$$

$$\nabla \mathrm{R}_{\mu\nu} - \frac{1}{2}\nabla \mathrm{g}_{\mu\nu} R = 0$$

$$\nabla\left(\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu} R\right) = 0 \tag{11}$$

So from eqn. 10 and 11

$$\nabla\left(\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu} R\right) = \nabla(8\pi G T_{\mu\nu}) \tag{12}$$

$$\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

For four dimensional purposes we have to divide right hand side of this eqn by C^4

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G T_{\mu\nu}}{C^4}$$

But Einstein realized he forgotten something, he remembered that, $\nabla g_{\mu\nu} = 0$. Therefore they could have include in eqn. 12 with any constant in front of it.

$$\nabla\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}\right) = \nabla\left(\frac{8\pi G T_{\mu\nu}}{C^4}\right)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G T_{\mu\nu}}{C^4}$$

First two term constitute the Einstein tensor. And Λ is called the cosmological constant, of which Einstein first thought of. When he was trying to identify how he could describe space in terms of mathematics

Reference: Lecture on General Relativity by Prof. Leonard Susskind, Stanford University, Oct-Dec-2008