Einstein Field Equation

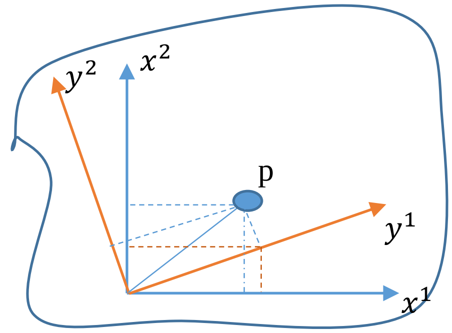


Fig. 1

Let us consider a coordinate system having axes $x^1, x^2, x^3 \ldots . x^n$. Then we can express the change in some function $\boldsymbol{\phi_s}$ in terms of these coordinates as

$$d\boldsymbol{\phi_s} = \sum_{\boldsymbol{n}} \frac{\partial \phi}{\partial x^n} d\boldsymbol{x^n} \quad (1)$$

We can use eqn. 1 with slight adjustment to show how a vector (tensor) transforms. Let a n dimension vector V in the y frame related with the m dimension vector V in the x frame as

$$V_y^n = \sum_m \frac{\partial y^n}{\partial x^m} V_x^m \quad (2)$$

Where m is a dummy variable.

Let us consider combination of m dimensional vector A^m and n dimensional vector B^n. Combination of this vector is a tensor T^{mn}.

$$T^{mn} = A^m B^n \quad (3)$$

In two dimensional space m and n can be either can have value 1 or 2, then T will have 4 versions. For 3 dimensional space, T^{mn} has 9 versions. Using eqn. 2, we can write,

$$T_y^{mn} = \sum_r \frac{\partial y^m}{\partial x^r} A_x^r \sum_s \frac{\partial y^n}{\partial x^s} B_x^s = \sum_{rs} \frac{\partial y^m}{\partial x^r} \frac{\partial y^n}{\partial x^s} A_x^r B_x^s$$

$$T_y^{mn} = \sum_{rs} \frac{\partial y^m}{\partial x^r} \frac{\partial y^n}{\partial x^s} T_x^{rs} \quad (4)$$

This transformation from y to x frame is called contravariant transformation. There is another transformation, covariant transformation which is,

$$T_{mn}^y = \sum_{rs} \frac{\partial x^r}{\partial y^m} \frac{\partial x^s}{\partial y^n} T_{rs}^x \quad (5)$$

According to Pythagoras,

$$ds^2 = dx^{1^2} + dx^{2^2} \ldots\ldots$$

$$= \sum_{mn} dx^m dx^n \, \delta_{mn} \quad (5a)$$

Where δ_{mn} is the ***Kronecker delta***, is a function of two variables, usually just non-negative integers. The function is 1 if the variables are equal, and 0 otherwise. Now rewriting the equation 1 in slightly different form as

$$dx^m = \sum_r \frac{\partial x^m}{\partial y^r} dy^r = \frac{\partial x^m}{\partial y^r} dy^r$$

Similarly, $$dx^n = \frac{\partial x^m}{\partial y^s} dy^s$$

Md. Saifur Rahman, Lecturer, Department of Physics, University of Rajshahi / 3H / General Relativity / 2019

Now substituting this value in eqn. 5a,

$$ds^2 = \delta_{mn} \sum_{mn} \frac{\partial x^m}{\partial y^r} dy^r \frac{\partial x^n}{\partial y^s} dy^s$$

$$= g_{mn} dy^r dy^s \tag{6}$$

Where $g_{mn} = \delta_{mn} \sum_{mn} \frac{\partial x^m}{\partial y^r} \frac{\partial x^m}{\partial y^s}$ is called the metric tensor. In flat space g_{mn} simply reduces to Kronecker delta δ_{mn} term. In the Einstein's field equation $g_{\mu\nu}$ terms refers the metric tensor which includes spacetime.

In case of curved surface we have to use the covariant derivatives. Covariant derivative of a tensor is equal to the ordinary derivative plus the correction term (Crystoffel symbol) multiplied by that tensor. Because the tensor $T_{mn}(y)$ equal to covariant derivative of vector V_m. Then

$$T_{mn}(y) = \nabla_n V_m = \frac{\partial V_m}{\partial y^n} + \Gamma^r_{nm} V_r(x) \tag{7}$$

Let us take the covariant derivative of the tensor T_{mn} for two indices from eqn. 7

$$\nabla_p T_{mn} = \frac{\partial T_{mn}}{\partial y^p} + \Gamma^r_{pm} T_{rn} + \Gamma^r_{pn} T_{mr} \tag{8}$$

From the property of tensor we can say that, this covariant derivative of metric tensor will be zero in all frame of reference. Substituting T_{mn} by g_{mn} in equation 8

$$\nabla_p g_{mn} = \frac{\partial g_{mn}}{\partial y^p} + \Gamma^r_{pm} g_{rn} + \Gamma^r_{pn} g_{rm} = 0$$

We can define a tangent vector, the rate of change of distance in a geodesic w.r.to proper time as $\frac{dx^\mu}{d\tau}$. In order for this distance is to be shortest, this derivative should be zero i.e. $\frac{dx^\mu}{d\tau} = 0$.

As we know, we have to use covariant derivative for curved space, let us take the covariant derivative of this tangent vector

$$\nabla \frac{dx^\mu}{d\tau} = \frac{\partial}{\partial \tau} \frac{\partial x^\mu}{\partial \tau} + \Gamma = 0$$

$$\frac{\partial^2 x^\mu}{\partial \tau^2} = -\Gamma = acceleration$$

From Newton's law, we know $a = \frac{F}{m} = -\Gamma$. Now we know the Cristoffel symbol

$$\Gamma^a_{bc}(x) = \frac{1}{2} g^{ad} \left\{ \frac{\partial g_{dc}}{\partial x^b} + \frac{\partial g_{ab}}{\partial x^c} - \frac{\partial g_{bc}}{\partial x^d} \right\} \tag{9}$$

The general relativity and Newton's law of gravitation must coincide for ordinary masses, gravity and speed. When that happens, g^{ad} becomes 1 and the derivatives becomes very very small, near to zero except the time component $\frac{\partial g_{00}}{\partial x}$ of the metric tensor.

So above eqn. becomes

$$\Gamma^a_{bc}(x) = \frac{1}{2}\frac{\partial g_{00}}{\partial x} \equiv F$$ For unit mass

But in Newtonian mechanics, force is negative gradient of potential, i.e.,

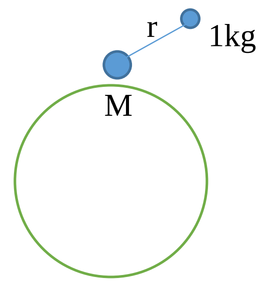
$$F = -\frac{d\varphi}{dx} \qquad \varphi = mgx$$ -ve sign indicates opposite direction of F and x

So the Cristoffel symbol

$$\Gamma^a_{bc}(x) = \frac{1}{2}\frac{\partial g_{00}}{\partial x} = \frac{d\varphi}{dx}$$

$$g_{00} = 2\varphi + constant \tag{9b}$$

Now the force capability across the whole sphere is



$$\int F.dA = -\frac{GM}{r^2}\int dA = -\frac{GM}{r^2}4\pi r^2 = -4\pi GM$$

Now from divergence theorem we can write,

$$\int \nabla.\boldsymbol{F}dV = -4\pi G\int \rho dV$$

$$\nabla.\boldsymbol{F} = -4\pi G\rho$$

$$\nabla^2\varphi = 4\pi G\rho$$

Earlier we shown that $g_{00} = 2\varphi + constant$. Let we drop the constant, then,

$$g_{00} = 2\varphi$$

$$\varphi = \frac{1}{2}g_{00}$$

So we get,

$$\nabla^2\frac{1}{2}g_{00} = 4\pi G\rho$$

$$\nabla^2 g_{00} = 8\pi G\rho \tag{9c}$$

Now the problem is, it is not a tensor equation, and for general relativity we need tensor equations. So what we looking for is something similar to it. So we want something that has the form $G_{\mu\nu}$, the Einstein's tensor, on the left side and $T_{\mu\nu}$, that contains all the mass, energy, pressure, stress terms, on the right side.

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{9d}$$

This is essentially a vector but we need tensor $T_{\mu\nu}$

T_{00} is the time component of the tensor
T_{01}, T_{02}, T_{03} are called energy flow paths of the tensor

T_{00}	T_{01}	T_{02}	T_{03}
T_{10}	T_{11}	T_{12}	T_{13}
T_{20}	T_{21}	T_{22}	T_{23}
T_{30}	T_{31}	T_{32}	T_{33}

T_{10}, T_{20}, T_{30} are called momentum density of the tensor , and rest 9 components are essentially momentum, flux, stress, pressure parts of the tensor

Now the right hand side of the eqn. (9d) is $8\pi G T_{\mu\nu}$ which is mass term, so left side must be curvature term. Einstein thought this can be Ricci curvature tensor.

$$R_{\mu\nu} = 8\pi G T_{\mu\nu}$$

But the problem is, energy neither be created nor be destroyed.

If we take the derivative $\partial T_{\mu\nu} = 0$

$$\partial R_{\mu\nu} \neq 0$$

We must always use the covariant derivative. So, if we got the covariant derivative of energy side is zero

$$\nabla(8\pi G T_{\mu\nu}) = 0 \tag{10}$$

$$\nabla T_{\mu\nu} = 0$$

we need something on the left side whose covariant derivative will also be zero. Einstein found that the covariant derivative of Ricci tensor is not zero but

$$\nabla \mathrm{R}_{\mu\nu} = \frac{1}{2}\nabla \mathrm{g}_{\mu\nu} R$$

$$\nabla\left(\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu} R\right) = 0 \tag{11}$$

So from eqn. 10 and 11

$$\nabla\left(\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu} R\right) = \nabla(8\pi G T_{\mu\nu}) \tag{12}$$

$$\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

For four dimensional purposes we have to divide right hand side of this eqn by C^4

$$\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu} R = \frac{8\pi G T_{\mu\nu}}{C^4}$$

But Einstein realized he forgotten something, he remembered that, $\nabla \mathrm{g}_{\mu\nu} = 0$. Therefore they could have include in eqn. 12 with any constant in front of it.

$$\nabla\left(\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu} R + \Lambda \mathrm{g}_{\mu\nu}\right) = \nabla\left(\frac{8\pi G T_{\mu\nu}}{C^4}\right)$$

$$\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu} R + \Lambda \mathrm{g}_{\mu\nu} = \frac{8\pi G T_{\mu\nu}}{C^4}$$

First two term constitute the Einstein tensor. And Λ is called the cosmological constant, of which Einstein first thought of. When he was trying to identify how he could describe space in terms of mathematics