Capacitors

Capacitance and Dielectrics

Consider two conductors carrying charges of equal magnitude but of opposite sign, as shown in Fig. 1. Such a combination of two conductors is called a capacitor. The conductors are called plates. A potential difference ΔV exists between the conductors due to the presence of the charges. Because the unit of potential difference is the volt, a potential difference is often called a voltage. We shall use this term to describe the potential difference across a circuit element or between two points in space.

What determines how much charge is on the plates of a capacitor for a given voltage? In other words, what is the *capacity* of the device for storing charge at a particular value of ΔV? Experiments show that the quantity of charge Q on a capacitor is linearly proportional to the potential difference between the conductors; that is, $Q \propto \Delta V$. The proportionality constant depends on the shape and separation of the conductors. We can write this relationship as $Q = C\Delta V$ if we define capacitance as follows:

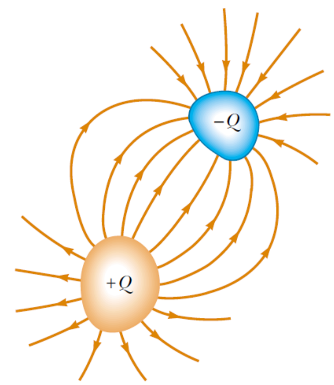

Fig. 1: A capacitor consists of two conductors carrying charges of equal magnitude but opposite sign.

The capacitance C of a capacitor is the ratio of the magnitude of the charge on either conductor to the magnitude of the potential difference between them:

$$C = \frac{Q}{\Delta V} \quad (1)$$

Capacitance has SI units of coulombs per volt. The SI unit of capacitance is the farad (F), which was named in honor of Michael Faraday: 1F=1C/V. The farad is a very large unit of capacitance. In practice, typical devices have capacitances ranging from microfarads (10^{-6} F) to picofarads (10^{-12} F). For practical purposes, capacitors often are labeled “mF” for microfarads and “mmF” for micromicrofarads or, equivalently, “pF” for picofarads.

Parallel-Plate Capacitors

Two parallel metallic plates of equal area A are separated by a distance d, as shown in Fig. 1. One plate carries a charge +Q, and the other carries a charge -Q. Let us consider how the geometry of these conductors influences the capacity of the combination to store charge. Recall that charges of like sign repel one another. As a capacitor is being charged by a battery, electrons flow into the negative plate and out of the positive plate. If the capacitor plates are large, the accumulated charges are able to distribute themselves over a substantial area, and the amount of charge that can be stored on a plate for a given potential difference increases as the plate area is increased. Thus, we expect the capacitance to be proportional to the plate area A.

Now let us consider the region that separates the plates. If the battery has a constant potential difference between its terminals, then the electric field between the plates must increase as d is decreased. Let us imagine that we move the plates closer together and consider the situation before any charges have had a chance to move in response to this change. Because no charges have moved, the electric field between the plates has the same value but extends over a shorter distance.

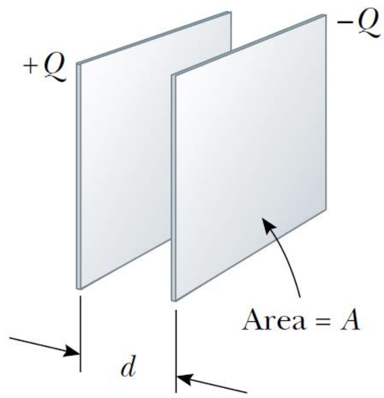

Fig. 1: A parallel-plate capacitor consists of two parallel conducting plates, each of area A, separated by a distance d.

Thus, the magnitude of the potential difference between the plates $\Delta V = Ed$ is now smaller. The difference between this new capacitor voltage and the terminal voltage of the battery now exists as a potential difference across the wires connecting the battery to the capacitor. This potential difference results in an electric field in the wires that drives more charge onto the plates, increasing the potential difference between the plates. When the potential difference between the plates again matches that of the battery, the potential difference across the wires falls back to zero, and the flow of charge stops. Thus, moving the plates closer together causes the charge on the capacitor to increase. If d is increased, the charge decreases. As a result, we expect the device's capacitance to be inversely proportional to d.

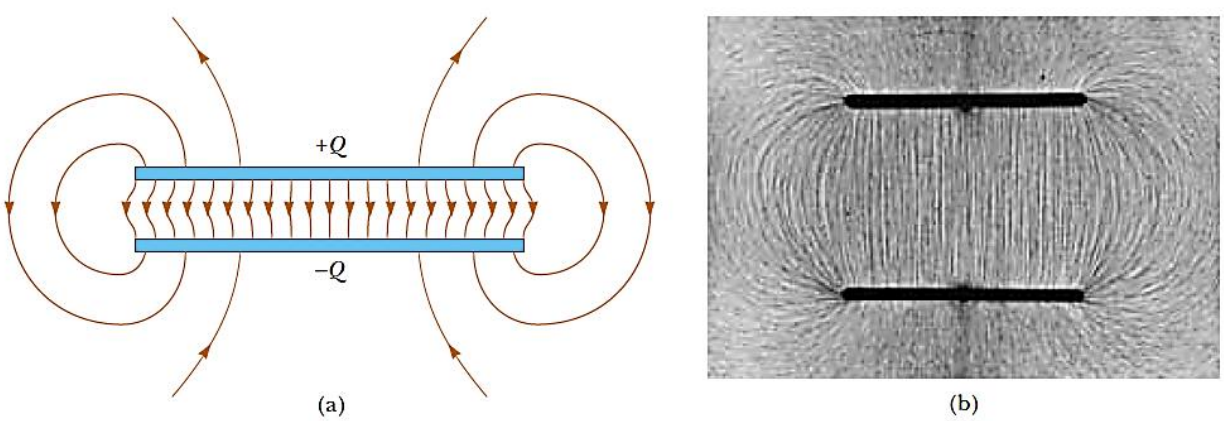

Fig. 2: (a) The electric field between the plates of a parallel-plate capacitor is uniform near the center but nonuniform near the edges. (b) Electric field pattern of two oppositely charged conducting parallel plates. Small pieces of thread on an oil surface align with the electric field.

We can verify these physical arguments with the following derivation. The surface charge density on either plate is $\sigma = \frac{Q}{A}$. If the plates are very close together (in comparison with their length and width), we can assume that the electric field is uniform between the plates and is zero elsewhere. The electric field between the plates is

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

Because the field between the plates is uniform, the magnitude of the potential difference between the plates equals Ed; therefore,

$$\Delta V = Ed = \frac{Qd}{\epsilon_0 A}$$

Substituting this result, we find that the capacitance is

$$C = \frac{Q}{\Delta V} = \frac{Q}{Qd/\epsilon_0 A}$$

$$C = \frac{\epsilon_0 A}{d}$$

That is, **the capacitance of a parallel-plate capacitor is proportional to the area of its plates and inversely proportional to the plate separation**, just as we expect from our conceptual argument.

The Cylindrical Capacitor

A solid cylindrical conductor of radius a and charge Q is coaxial with a cylindrical shell of negligible thickness, radius $b > a$ and charge - Q. Find the capacitance of this cylindrical capacitor if its length is l.

Solution: It is difficult to apply physical arguments to this configuration, although we can reasonably expect the capacitance to be proportional to the cylinder length l for the same reason that parallel-plate capacitance is proportional to plate area: Stored charges have more room in which to be distributed. If we assume that l is much greater than a and b, we can neglect end effects. In this case, the electric field is perpendicular to the long axis of the cylinders and is confined to the region between them (Fig. 1b). We must first calculate the potential difference between the two cylinders, which is given in general by

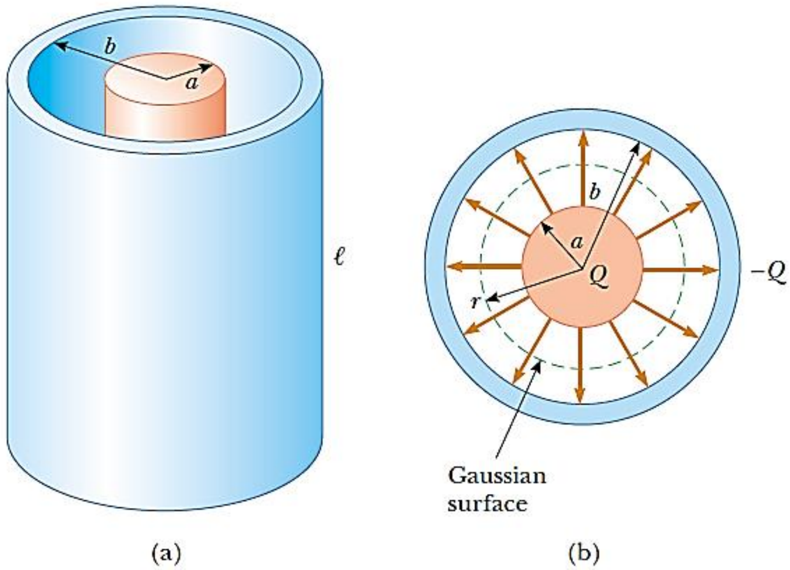

Fig. 1: (a) A cylindrical capacitor consists of a solid cylindrical conductor of radius a and length *l* surrounded by a coaxial cylindrical shell of radius b. (b) End view. The dashed line represents the end of the cylindrical Gaussian surface of radius r and length *l*.

$$V_b - V_a = -\int_a^b \mathbf{E} \cdot d\mathbf{s}$$

Where E is the electric field in the region a < r < b. We showed using Gauss's law that the magnitude of the electric field of a cylindrical charge distribution having linear charge density λ is $E_r = 2K_e\frac{\lambda}{r}$. The same result applies here because, according to Gauss's law, the charge on the outer cylinder does not contribute to the electric field inside it. Using this result and noting from Fig.1b that E is along r, we find that

$$V_b - V_a = -\int_a^b E_r\, dr = -2k_e\lambda \int_a^b \frac{dr}{r} = -2k_e\lambda \ln\left(\frac{b}{a}\right)$$

Substituting this result into Eqn. 33 and using the fact that $\lambda = \frac{Q}{l}$, we obtain

$$C = \frac{Q}{\Delta V} = \frac{Q}{\dfrac{2k_e Q}{\ell}\ln\left(\dfrac{b}{a}\right)} = \frac{\ell}{2k_e \ln\left(\dfrac{b}{a}\right)} \qquad (1)$$

Where ΔV is the magnitude of the potential difference, given by $\Delta V = |V_b - V_a| = 2k_e\lambda\, ln(b/a)$, a positive quantity. As predicted, the capacitance is proportional to the length of the cylinders. As we might expect, the capacitance also depends on the radii of the two cylindrical conductors. From Eqn 1, we see that the capacitance per unit length of a combination of concentric cylindrical conductors is

$$\frac{C}{\ell} = \frac{1}{2k_e \ln\left(\frac{b}{a}\right)}$$

An example of this type of geometric arrangement is a coaxial cable, which consists of two concentric cylindrical conductors separated by an insulator. The cable carries electrical signals in the inner and outer conductors. Such a geometry is especially useful for shielding the signals from any possible external influences.

The Spherical Capacitor

A spherical capacitor consists of a spherical conducting shell of radius b and charge -*Q* concentric with a smaller conducting sphere of radius a and charge *Q*. Find the capacitance of this device

Solution:

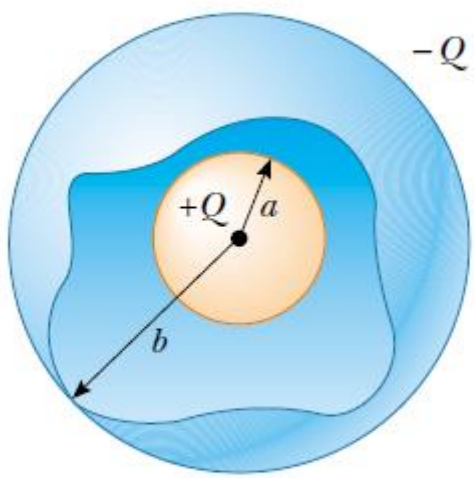

Fig. 1: A spherical capacitor consists of an inner sphere of radius a surrounded by a concentric spherical shell of radius b. The electric field between the spheres is directed radially outward when the inner sphere is positively charged.

As we know that, the field outside a spherically symmetric charge distribution is radial and given by the expression $E=K_eQ/r^2$. In this case, this result applies to the field between the spheres $(a < r < b)$. From Gauss's law we see that only the inner sphere contributes to this field. Thus, the potential difference between the spheres is

$$V_b - V_a = -\int_a^b E_r\,dr = -k_eQ\int_a^b \frac{dr}{r^2} = k_eQ\left[\frac{1}{r}\right]_a^b$$

$$= k_eQ\left(\frac{1}{b} - \frac{1}{a}\right)$$

The magnitude of the potential difference is

$$\Delta V = |V_b - V_a| = k_eQ\frac{(b-a)}{ab}$$

Substituting this value for ΔV into Eqn 33, we obtain

$$C = \frac{Q}{\Delta V} = \frac{ab}{k_e(b-a)}$$

Parallel Combination of Capacitors:

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Two capacitors connected as shown in Fig. 1 are known as a parallel combination of capacitors. The left plates of the capacitors are connected by a conducting wire to the positive terminal of the battery and are therefore both at the same electric potential as the positive terminal. Likewise, the right plates are connected to the negative terminal and are therefore both at the same potential as the negative terminal. Thus, **the individual potential differences across capacitors connected in parallel are all the same and are equal to the potential difference applied across the combination.** In a circuit such as that shown in Figure 1, the voltage applied across the combination is the terminal voltage of the battery. Situations can occur in which the parallel combination is in a circuit with other circuit elements; in such situations, we must determine the potential difference across the combination by analyzing the entire circuit.

When the capacitors are first connected in the circuit shown in Figure 1, electrons are transferred between the wires and the plates; this transfer leaves the left plates positively charged and the right plates negatively charged. The energy source for this charge transfer is the internal chemical energy stored in the battery, which is converted to electric potential energy associated with the charge separation. The flow of charge ceases when the voltage across the capacitors is equal to that across the battery terminals. The capacitors reach their maximum charge when the flow of charge ceases. Let us call the maximum charges on the two capacitors Q_1 and Q_2 . The total charge Q stored by the two capacitors is

$$Q = Q_1 + Q_2 \qquad (1)$$

That is, **the total charge on capacitors connected in parallel is the sum of the charges on the individual capacitors**. Because the voltages across the capacitors are the same, the charges that they carry are

$$Q_1 = C_1 \Delta V \qquad\qquad Q_2 = C_2 \Delta V$$

Suppose that we wish to replace these two capacitors by one equivalent capacitor having a capacitance Ceq , as shown in Figure 1c. The effect this equivalent capacitor has on the circuit must be exactly the same as the effect of the combination of the two individual capacitors. That is, the equivalent capacitor must store Q units of charge when connected to the battery. We can see from Fig. 1c that the voltage across the equivalent capacitor also is ΔV because the equivalent capacitor is connected directly across the battery terminals. Thus, for the equivalent capacitor,

$$Q = C_{eq}\, \Delta V$$

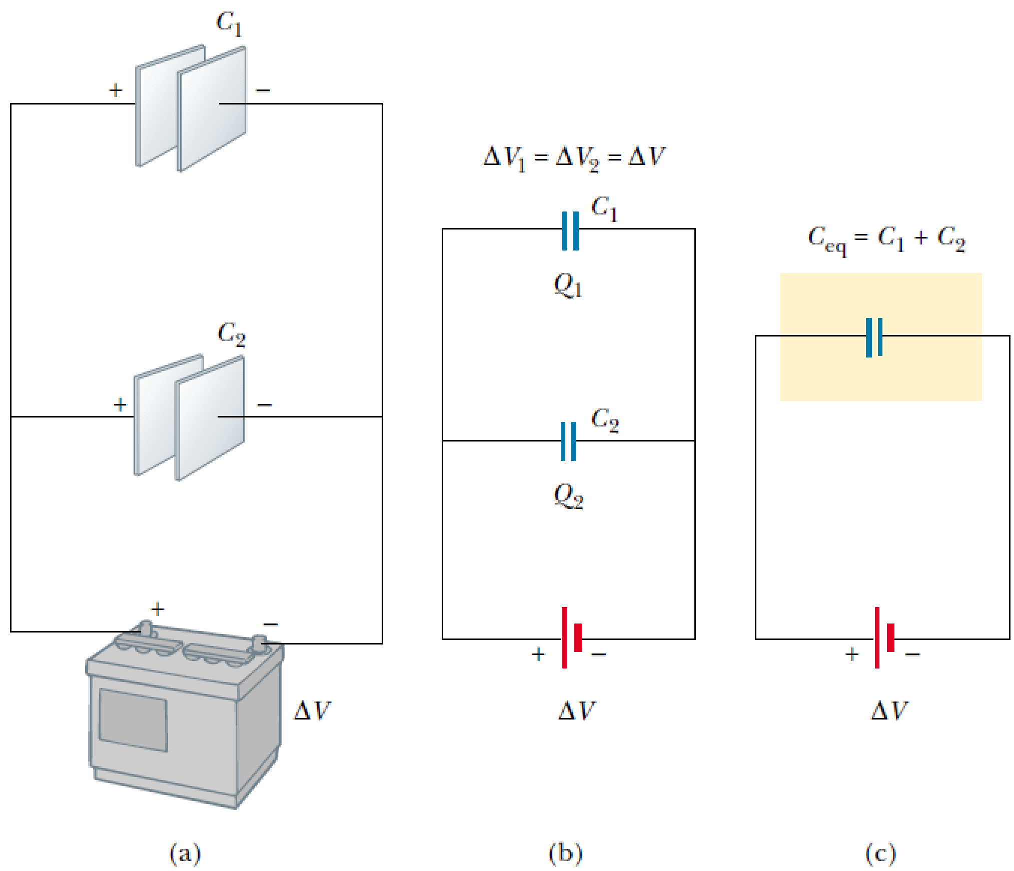

Fig. 1: (a) A parallel combination of two capacitors in an electric circuit in which the potential difference across the battery terminals is ΔV. (b) The circuit diagram for the parallel combination. (c) The equivalent capacitance is $C_{eq} = C_1 + C_2$

Substituting these three relationships for charge into Eqn. 35, we have

$$C_{eq}\, \Delta V = C_1\, \Delta V + C_2\, \Delta V$$

$$C_{eq} = C_1 + C_2 \qquad \begin{pmatrix} \text{parallel} \\ \text{combination} \end{pmatrix}$$

If we extend this treatment to three or more capacitors connected in parallel, we find the equivalent capacitance to be

$$C_{eq} = C_1 + C_2 + C_3 + \cdots \qquad \text{(parallel combination)}$$

Thus, **the equivalent capacitance of a parallel combination of capacitors is greater than any of the individual capacitances.** This makes sense because we are essentially combining the areas of all the capacitor plates when we connect them with conducting wire.

Series Combination

Two capacitors connected as shown in Figure 1a are known as a series combination of capacitors. The left plate of capacitor 1 and the right plate of capacitor 2 are connected to the terminals of a battery. The other two plates are connected to each other and to nothing else; hence, they form an isolated conductor that is initially uncharged and must continue to have zero net charge. To analyze this combination, let us begin by considering the uncharged capacitors and follow what happens just after a battery is connected to the circuit. When the battery is connected, electrons are transferred out of the left plate of C_1 and into the right plate of C_2. As this negative charge accumulates on the right plate of C_2, an equivalent amount of negative charge is forced off the left plate of C_2, and this left plate therefore has an excess positive charge. The negative charge leaving the left plate of C2 travels through the connecting wire and accumulates on the right plate of C_1. As a result, all the right plates end up with a charge - Q, and all the left plates end up with a charge +Q. Thus, **the charges on capacitors connected in series are the same.**

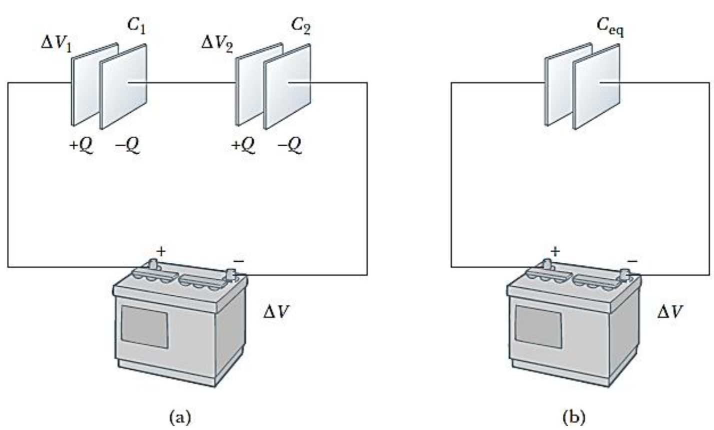

Fig. 1: (a) A series combination of two capacitors. The charges on the two capacitors are the same. (b) The capacitors replaced by a single equivalent capacitor.

From Figure 1a, we see that the voltage ΔV across the battery terminals is split between the two capacitors:

$$\Delta V = \Delta V_1 + \Delta V_2$$

Where ΔV_1 and ΔV_2 are the potential differences across capacitors C_1 and C_2, respectively. In general, the total potential difference across any number of capacitors connected in series is the sum of the potential differences across the individual capacitors. Suppose that an equivalent capacitor has the same effect on the circuit as the series combination. After it is fully charged, the equivalent capacitor must have a charge of -Q on its right plate and a charge of +Q on its left plate. Applying the definition of capacitance to the circuit in Figure 1b, we have

$$\Delta V = \frac{Q}{C_{\text{eq}}}$$

Because we can apply the expression $Q=C\Delta V$ to each capacitor shown in Figure 1a, the potential difference across each is

Energy Stored in a Charged Capacitor

Consider a parallel-plate capacitor that is initially uncharged, such that the initial potential difference across the plates is zero. Now imagine that the capacitor is connected to a battery and develops a maximum charge Q. (We assume that the capacitor is charged slowly so that the problem can be considered as an electrostatic system.) When the capacitor is connected to the battery, electrons in the wire just outside the plate connected to the negative terminal move into the plate to give it a negative charge. Electrons in the plate connected to the positive terminal move out of the plate into the wire to give the plate a positive charge. Thus, charges move only a small distance in the wires. To calculate the energy of the capacitor, we shall assume a different process - one that does not actually occur but gives the same final result. We can make this assumption because the energy in the final configuration does not depend on the actual charge-transfer process. We imagine that we reach in and grab a small amount of positive charge on the plate connected to the negative terminal and apply a force that causes this positive charge to move over to the plate connected to the positive terminal. Thus, we do work on the charge as we transfer it from one plate to the other. At first, no work is required to transfer a small amount of charge dq from one plate to the other. However, once this charge has been transferred, a small potential difference exists between the plates. Therefore, work must be done to move additional charge through this potential difference. As more and more charge is transferred from one plate to the other, the potential difference increases in proportion, and more work is required.

Suppose that q is the charge on the capacitor at some instant during the charging process. At the same instant, the potential difference across the capacitor is $\Delta V = q/C$. We know that the work necessary to transfer an increment of charge dq from the plate carrying charge $-q$ to the plate carrying charge q (which is at the higher electric potential) is

$$dW = \Delta V\, dq = \frac{q}{C}\, dq$$

The total work required to charge the capacitor from $q=0$ to some final charge $q=Q$ is

$$W = \int_0^Q \frac{q}{C}\, dq = \frac{1}{C}\int_0^Q q\, dq = \frac{Q^2}{2C}$$

The work done in charging the capacitor appears as electric potential energy U stored in the capacitor. Therefore, we can express the potential energy stored in a charged capacitor in the following forms:

$$U = \frac{Q^2}{2C} = \tfrac{1}{2}Q\,\Delta V = \tfrac{1}{2}C(\Delta V)^2 \tag{1}$$

This result applies to any capacitor, regardless of its geometry. We see that for a given capacitance, the stored energy increases as the charge increases and as the potential difference increases. In practice, there is a limit to the maximum energy (or charge) that can be stored because, at a sufficiently great value of ΔV, discharge ultimately occurs between the plates. For this reason, capacitors are usually labeled with a maximum operating voltage.

For a parallel-plate capacitor, the potential difference is related to the electric field through the relationship ΔV = Ed. Furthermore, its capacitance is $C = \epsilon_0 A/d$. Substituting these expressions into Equation 1, we obtain

$$U = \frac{1}{2}\frac{\epsilon_0 A}{d}(E^2 d^2) = \frac{1}{2}(\epsilon_0 A d)E^2 \tag{2}$$

Because the volume V (volume, not voltage!) occupied by the electric field is Ad, the energy per unit volume $U_E = \frac{U}{V} = \frac{U}{Ad}$ known as the energy density, is

$$u_E = \tfrac{1}{2}\epsilon_0 E^2 \tag{3}$$

Although Equation 3 was derived for a parallel-plate capacitor, the expression is generally valid. That is, **the energy density in any electric field is proportional to the square of the magnitude of the electric field at a given point.**

Capacitors with Dielectrics

A **dielectric** is a nonconducting material, such as rubber, glass, or waxed paper. When a dielectric is inserted between the plates of a capacitor, the capacitance increases. If the dielectric completely fills the space between the plates, the capacitance increases by a dimensionless factor K, which is called the **dielectric constant**. The dielectric constant is a property of a material and varies from one material to another. In this section, we analyze this change in capacitance in terms of electrical parameters such as electric charge, electric field, and potential difference.

We can perform the following experiment to illustrate the effect of a dielectric in a capacitor: Consider a parallel-plate capacitor that without a dielectric has a charge Q_0 and a capacitance C_0. The potential difference across the capacitor is $\Delta V_0 = \frac{Q_0}{C_0}$. Figure 1a illustrates this situation. The potential difference is measured by a voltmeter.

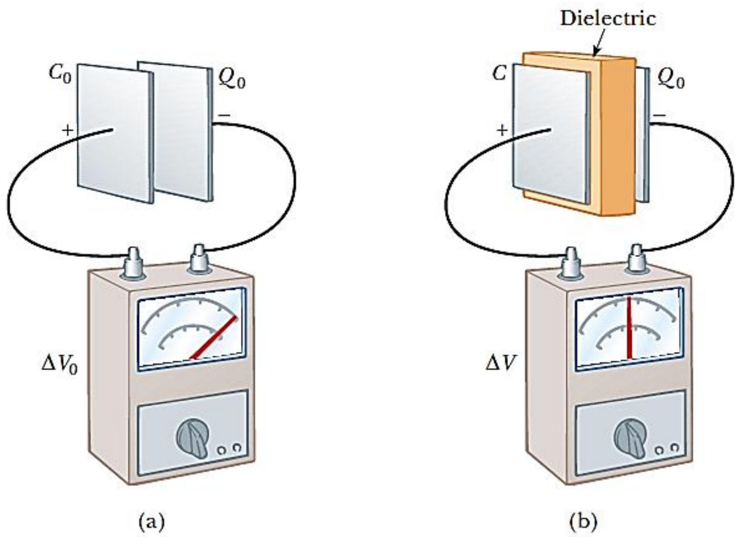

Fig.1: Charged capacitor (a) before and (b) after insertion of a dielectric between the plates.

Note that no battery is shown in the figure; also, we must assume that no charge can flow through an ideal voltmeter. Hence, there is no path by which charge can flow and alter the charge on the capacitor. If a dielectric is now inserted between the plates, as shown in Figure 1b, the voltmeter indicates that the voltage between the plates decreases to a value ΔV. The voltages with and without the dielectric are related by the factor *K* as follows:

$$\Delta V = \frac{\Delta V_0}{\kappa}$$

Because $\Delta V < \Delta V_0$, we see that k>1. Because the charge Q_0 on the capacitor does not change, we conclude that the capacitance must change to the value

$$C = \frac{Q_0}{\Delta V} = \frac{Q_0}{\Delta V_0/\kappa} = \kappa \frac{Q_0}{\Delta V_0}$$

$$C = \kappa C_0 \tag{1}$$

That is, the capacitance increases by the factor *K* when the dielectric completely fills the region between the plates. For a parallel-plate capacitor, where $C=\frac{\epsilon_0 A}{d}$, we can express the capacitance when the capacitor is filled with a dielectric as

$$C=k\frac{\epsilon_0 A}{d} \tag{2}$$

From Equation 2, it would appear that we could make the capacitance very large by decreasing d, the distance between the plates. In practice, the lowest value of d is limited by the electric discharge that could occur through the dielectric medium separating the plates. For any given separation d, the maximum voltage that can be applied to a capacitor without causing a discharge depends on the dielectric strength (maximum electric field) of the dielectric. If the magnitude of the electric field in the dielectric exceeds the dielectric strength, then the insulating properties break down and the dielectric begins to conduct. Insulating materials have values of k greater than unity and dielectric strengths greater than that of air, as Table 1 indicates. Thus, we see that a dielectric provides the following advantages:

- Increase in capacitance
- Increase in maximum operating voltage
- Possible mechanical support between the plates, which allows the plates to be close together without touching, thereby decreasing d and increasing C

Table 1: Dielectric Constants and Dielectric Strengths of Various Materials at Room Temperature Dielectric

Material	Dielectric Constant κ	Dielectric Strength[a] (V/m)
Air (dry)	1.000 59	3×10^6
Bakelite	4.9	24×10^6
Fused quartz	3.78	8×10^6
Neoprene rubber	6.7	12×10^6
Nylon	3.4	14×10^6
Paper	3.7	16×10^6
Polystyrene	2.56	24×10^6
Polyvinyl chloride	3.4	40×10^6
Porcelain	6	12×10^6
Pyrex glass	5.6	14×10^6
Silicone oil	2.5	15×10^6
Strontium titanate	233	8×10^6
Teflon	2.1	60×10^6
Vacuum	1.000 00	—
Water	80	—

An Atomic Description of Dielectrics

We know that the potential difference ΔV_0 between the plates of a capacitor is reduced to $\Delta V_0/k$ when a dielectric is introduced. Because the potential difference between the plates equals the product of the electric field and the separation d, the electric field is also reduced. Thus, if $\mathbf{E_0}$ is the electric field without the dielectric, the field in the presence of a dielectric is

$$\mathbf{E}=\frac{\boldsymbol{E_o}}{\boldsymbol{k}} \qquad (1)$$

Let us first consider a dielectric made up of polar molecules placed in the electric field between the plates of a capacitor. The dipoles (that is, the polar molecules making up the dielectric) are randomly oriented in the absence of an electric field, as shown in Figure 1a. When an external field $\mathbf{E_0}$ due to charges on the capacitor plates is applied, a torque is exerted on the dipoles, causing them to partially align with the field, as shown in Figure 1b. We can now describe the dielectric as being polarized. The degree of alignment of the molecules with the electric field depends on temperature and on the magnitude of the field. In general, the alignment increases with decreasing temperature and with increasing electric field.

If the molecules of the dielectric are nonpolar, then the electric field due to the plates produces some charge separation and an *induced dipole moment*. These induced dipole moments tend to align with the external field, and the dielectric is polarized. Thus, we can polarize a dielectric with an external field regardless of whether the molecules are polar or nonpolar.

With these ideas in mind, consider a slab of dielectric material placed between the plates of a capacitor so that it is in a uniform electric field $\mathbf{E_0}$, as shown in Figure 2a. The electric field due to the plates is directed to the right and polarizes the dielectric. The net effect on the dielectric is the formation of an induced positive surface charge density $+\sigma_{ind}$ on the right face and an equal negative surface charge density $-\sigma_{ind}$ on the left face, as shown in Figure 2b. These induced

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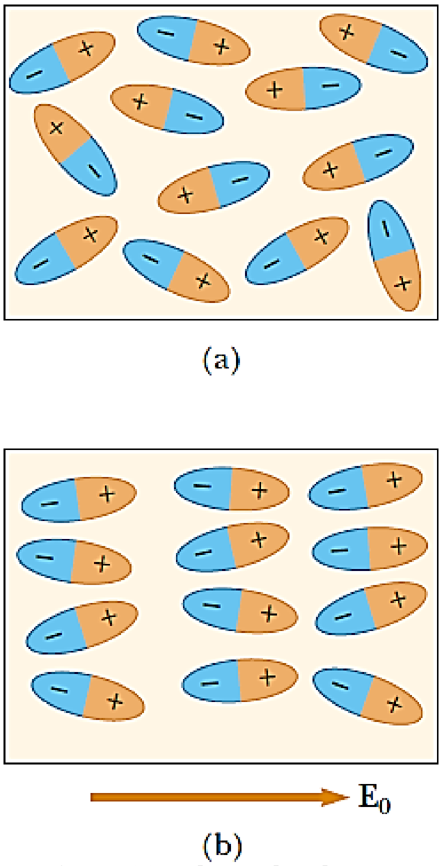


Fig. 1: (a) Polar molecules are randomly oriented in the absence of an external electric field. (b) When an external field is applied, the molecules partially align with the field.

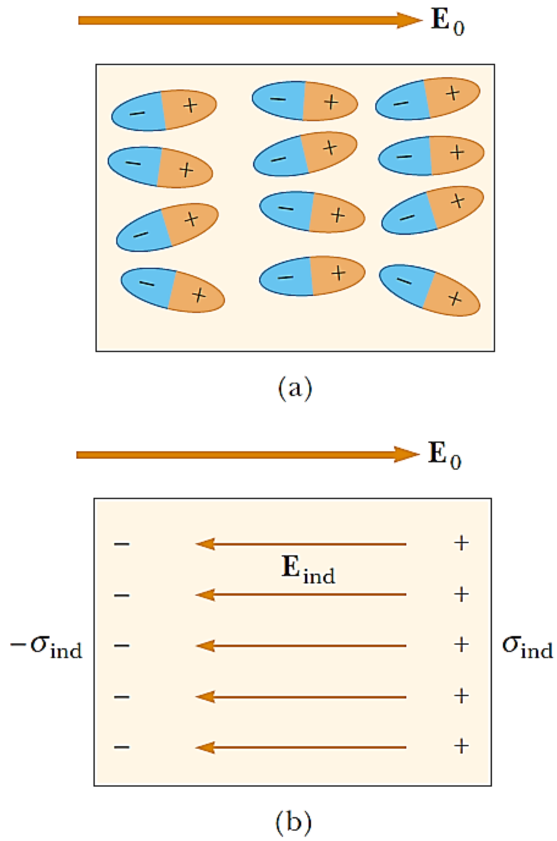


Fig. 2: (a) When a dielectric is polarized, the dipole moments of the molecules in the dielectric are partially aligned with the external field E_0. (b) This polarization causes an induced negative surface charge on one side of the dielectric and an equal induced positive surface charge on the opposite side. This separation of charge results in a reduction in the net electric field within the dielectric.

surface charges on the dielectric give rise to an induced electric field E_{ind} in the direction opposite the external field $\mathbf{E_0}$. Therefore, the net electric field **E** in the dielectric has a magnitude

$$E = E_0 - E_{ind} \qquad (2)$$

In the parallel-plate capacitor shown in Figure 3, the external field $\mathbf{E_0}$ is related to the charge density σ on the plates through the relationship $E = \frac{\sigma}{\epsilon_0}$. The induced electric field in the dielectric is related to the induced charge density σ_{ind} through the relationship $E_{ind} = \frac{\sigma_{ind}}{k}$. Because $E=E_0/k= \sigma / \epsilon_0$ substitution into Equation 2 gives

$$\frac{\sigma}{\kappa\epsilon_0} = \frac{\sigma}{\epsilon_0} - \frac{\sigma_{ind}}{\epsilon_0}$$

$$\sigma_{ind} = \left(\frac{\kappa - 1}{\kappa}\right)\sigma$$

Because k>1 this expression shows that the charge density σ_{ind} induced on the dielectric is less than the charge density σ on the plates. For instance, if k=3 we see that the induced charge density is two-thirds the charge density on the plates. If no dielectric is present, then k=1 and $\sigma_{ind} = 0$ as expected. However, if the dielectric is replaced by an electrical conductor, for which E=0 then Eqn.2 indicates that this corresponds to $\sigma_{ind} = \sigma$. That is, the surface charge induced on the conductor is equal in magnitude but opposite in sign to that on the plates, resulting in a net electric field of zero in the conductor.

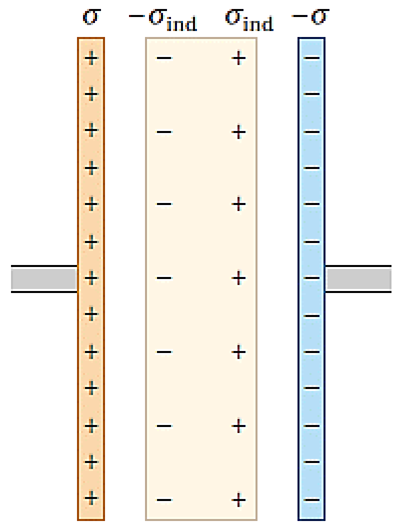


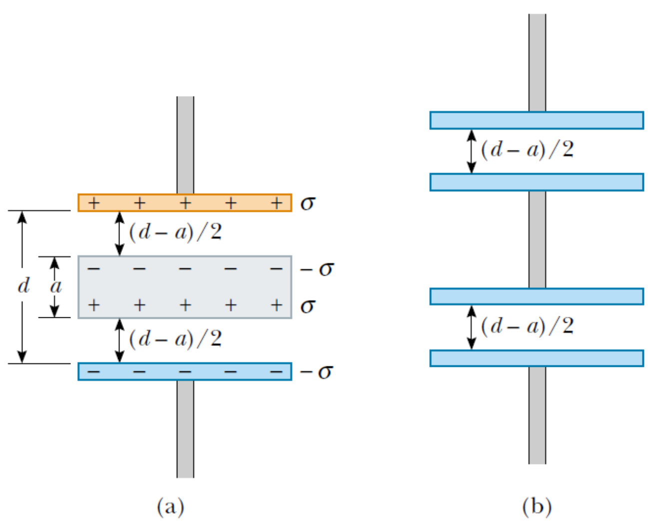
Fig. 29: Induced charge on a dielectric placed between the plates of a charged capacitor. Note that the induced charge density on the dielectric is less than the charge density on the plates.

Effect of a Metallic Slab

Problem 1: A parallel-plate capacitor has a plate separation d and plate area A. An uncharged metallic slab of thickness a is inserted midway between the plates. (a) Find the capacitance of the device. (b) Show that the capacitance is unaffected if the metallic slab is infinitesimally thin. (c) Show that the answer to part (a) does not depend on where the slab is inserted.

Solution:

(a): We can solve this problem by noting that any charge that appears on one plate of the capacitor must induce a charge of equal magnitude but opposite sign on the near side of the slab, as shown in Figure 1a. Consequently, the net charge on the slab remains zero, and the electric field inside the slab is zero. Hence, the capacitor is equivalent to two capacitors in series, each having a plate separation $\frac{d-a}{2}$ as shown in Figure 1b. Using the rule for adding two capacitors in series, we obtain A Partially Filled Capacitor



$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{\frac{\epsilon_0 A}{(d-a)/2}} + \frac{1}{\frac{\epsilon_0 A}{(d-a)/2}}$$

$$C = \frac{\epsilon_0 A}{d - a}$$

(b) In the result for part (a), we let a → 0:

$$C = \lim_{a\to 0} \frac{\epsilon_0 A}{d - a} = \frac{\epsilon_0 A}{d}$$

Which is the original capacitance. So, the capacitance is unaffected if the inserted metallic slab is infinitesimally thin.

(c) Let us imagine that the slab in Figure 1a is moved upward so that the distance between the upper edge of the slab and the upper plate is b. Then, the distance between the lower edge of the slab and the lower plate is $d - b - a$. As in part (a), we find the total capacitance of the series combination:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{\frac{\epsilon_0 A}{b}} + \frac{1}{\frac{\epsilon_0 A}{d-b-a}}$$

$$= \frac{b}{\epsilon_0 A} + \frac{d-b-a}{\epsilon_0 A} = \frac{d-a}{\epsilon_0 A}$$

$$C = \frac{\epsilon_0 A}{d - a}$$

This is the same result as in part (a). It is independent of the value of b, so it does not matter where the slab is located.

A Partially Filled Capacitor

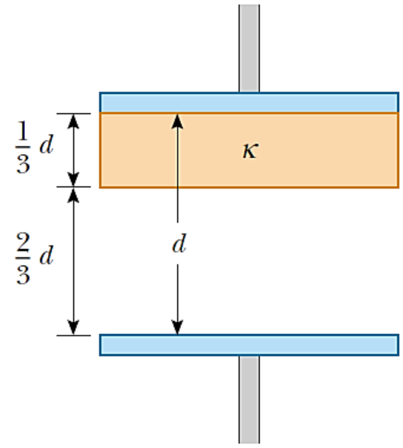
Problem 2:

A parallel-plate capacitor with a plate separation d has a capacitance C_0 in the absence of a dielectric. What is the capacitance when a slab of dielectric material of dielectric constant k and thickness d/3 is inserted between the plates?

Solution

In Example 1, we found that we could insert a metallic slab between the plates of a capacitor and consider the combination as two capacitors in series. The resulting capacitance was independent of the location of the slab.

Furthermore, if the thickness of the slab approaches zero, then the capacitance of the system approaches the capacitance when the slab is absent. From this, we conclude that we can insert an infinitesimally thin metallic slab anywhere between the plates of a capacitor without affecting the capacitance. Thus, let us imagine sliding an infinitesimally thin metallic slab along the bottom face of the dielectric shown in Figure 1a. We can then consider this system to be the series combination of the two capacitors shown in Figure 1b: one having a plate separation d/3 and filled with a dielectric, and the other having a plate separation 2d/3 and air between its plates.



(a)

From Equation of paralallel plate capacitor, the two capacitances are

$$C_1 = \frac{\kappa\epsilon_0 A}{d/3} \quad \text{and} \quad C_2 = \frac{\epsilon_0 A}{2d/3}$$

For two capacitors combined in series, we have

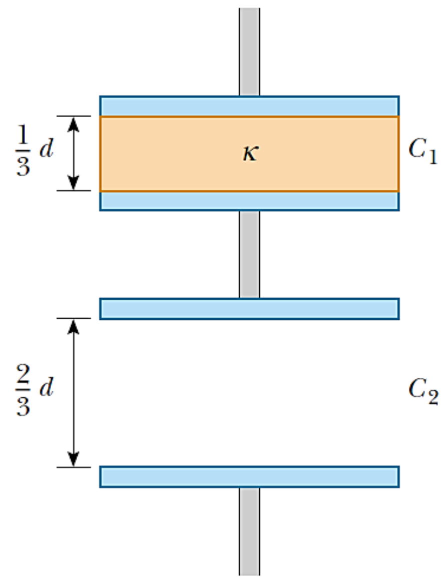
$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{d/3}{\kappa\epsilon_0 A} + \frac{2d/3}{\epsilon_0 A}$$

$$= \frac{d}{3\epsilon_0 A}\left(\frac{1}{\kappa} + 2\right) = \frac{d}{3\epsilon_0 A}\left(\frac{1+2\kappa}{\kappa}\right)$$

$$C = \left(\frac{3\kappa}{2\kappa+1}\right)\frac{\epsilon_0 A}{d}$$

Because the capacitance without the dielectric is $C = \frac{\varepsilon_0 A}{d}$ we see that

$$C = \left(\frac{3\kappa}{2\kappa+1}\right)C_0$$



(b)

Polarization:

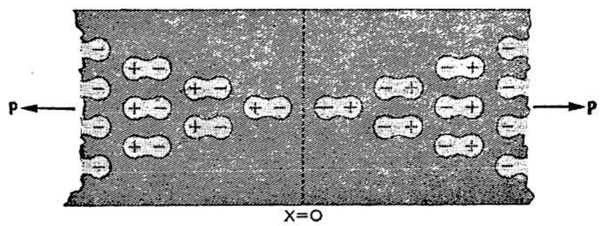
Let the small displacement of the charge q_i in an induced dipole be l_i. Then the electric moment of the dipole is $\vec{p}_i = q_i\vec{l}_i$. Now, if there are n dipoles in a small volume ΔV around a given point, then the polarization at that point is defined as the vector sum of the n dipoles divided by the volume ΔV. In other words,

$$\vec{P} = \frac{1}{\Delta V}\sum_{i=1}^{n}\vec{p}_i = N\vec{p}$$

Where N is the number of dipoles per unit volume and $\vec{p}$ is the average dipole moment. The polarization $\vec{P}$ is, therefore, equal to the resultant dipole moment per unit volume. When the medium is strongly polarized, all the dipole moments in ΔV will be practically parallel and the vector sum of them will nearly be equal to the arithmetic sum and the resultant $\vec{P}$ will be large. But if the field is weak there can be two things. Either the magnitude of $\vec{p}_i$ of individual dipole mement is very small or the dipoles are oriented at random. In both these cases the resultant $\vec{P}$ is very small or practically zero.

Polarization Charges:

Let us first consider a uniformly polarized medium with polarization $\vec{P}$. Within the volume of this medium the positive end of one dipole will be next to the negative end of a neighboring dipole, so that there will be no net volume charge, Fig 1. Let us consider a surface of area A, whose normal is in the direction of $\vec{P}$.



This surface forms a volume *Al* extending a distance *l* into the medium, where *l* is the length of a dipole. The electric moment of this volume will be *AlP*. The charge *q* that must be displaced a distance *l* from the charge -q to produce this dipole moment is, then,

$$\vec{p} = \vec{l}q = Al\vec{P} = A\vec{l}P$$

$$q = AP \qquad (1)$$

On the surface there will appear an unbalanced positive charge per unit area which is equal to P. We denote the surface density due to polarization by σ_p, so that

$$\sigma_p = \vec{P}.\hat{n} = P$$

Where $\hat{n}$ is a unit vector in the direction of the outward normal. Next, we suppose that the polarization $\vec{P}$ is not uniform, but varies throughout the medium, Fig. 1. Let S be a surface enclosing a volume v of the medium. Then, the surface integral of $\vec{P}.d\vec{a}$ will represent the charge q_p leaving *V* across the surface *S* during polarization of the medium, that is,

$$q_p = \oint_S \vec{P}.d\vec{a} \qquad (2)$$

There is a volume polarization density ρ_p. From the principle of conservation of charge, a net charge q_p leaving a volume that was originally neutral must leave a charge - q_p inside the volume, where

$$-q_p = \int_V \rho_p . dV \qquad (3)$$

Combining Eqs. (2) and (3), and applying the divergence theorem we get,

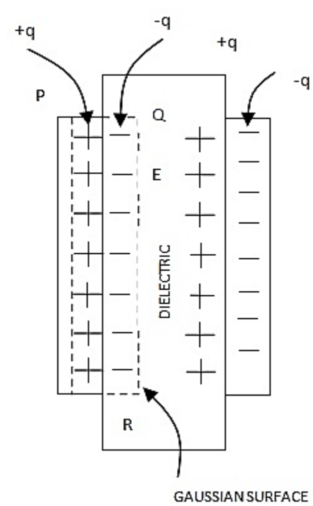
$$\oint_S \vec{P} . d\vec{a} = -\int_V \rho_p . dV$$

$$\int_V \vec{\nabla} . \vec{P} dV = -\int_V \rho_p . dV$$

or, equating the integrands

$$\rho_p = -\vec{\nabla} . \vec{P} \qquad (4)$$

Gauss's law in dielectrics:



Suppose a material of dielectric constant K is introduced in the intervening space between the two plates. The dielectric slab gets polarized. A negative charged – q and + q on the dielectric are called the 'induced charges' or 'bound charges' while the charges +q and – q on the capacitor plates are called free charges. These induced charges produce their own field which opposes the external field $\vec{E}_0$. Let $\vec{E}$ be the resultant field within the dielectric. The net charge within the Gaussian surface is q – q.

Let us apply Gauss' law to the surface S enclosing the volume V. To extend this law to media other than free space we must write

$$\oint_S \varepsilon_0 \vec{E} . d\vec{a} = \int_V (\rho + \rho_p) dV \qquad (1)$$

The inclusion of pp takes into account the effect of polarization of the medium. As we know $\rho_p = -\vec{\nabla} . \vec{P}$, then

$$\oint_S \varepsilon_0 \vec{E} . d\vec{a} = \int_V (\rho - \vec{\nabla} . \vec{P}) dV$$

(2)

$$\oint_S \varepsilon_0\vec{E}.d\vec{a} = \int_V \rho dV - \int_V \vec{\nabla}.\vec{P}dV$$

The last term can be transformed into a surface integral by divergence theorem, so that

$$\oint_S \varepsilon_0\vec{E}.d\vec{a} = \int_V \rho dV - \oint_S \vec{P}dV$$

$$\oint_S (\varepsilon_0\vec{E} + \vec{P}).d\vec{a} = \int_V \rho dV \tag{3}$$

We define the electric displacement $\overrightarrow{D}$ for the dielectric medium as

$$\overrightarrow{D} = \varepsilon_0\vec{E} + \vec{P} \tag{4}$$

For air or vacuum $\vec{P} = 0$ so $\overrightarrow{D} = \varepsilon_0\vec{E}$. Eq. (3) now becomes

$$\oint_S \overrightarrow{D}.d\vec{a} = \int_V \rho dV \tag{5}$$

which is Gauss' law for dielectric. The right side of Eq. (5) is the total free charge within the volume *V* of the dielectric. By divergence theorem, the left side of Eq. (5) can also be transformed into a volume integral. Thus

$$\int_V \vec{\nabla}.\vec{D}dV = \int_V \rho dV$$

or, equating the integrands, we find

$$\vec{\nabla}.\vec{D} = \rho$$

Which is sometimes referred to as the differential form of Gauss' law.

We have eliminated the polarization charge density pp, but we have introduced D instead of E. Hence, to determine E in the dielectric we must know the relation between D and E. This relation will be found in the next section. Eq. (4) cannot be used, because P depends on E also and makes complications.

***Isotropic Dielectric**: A dielectric whose polarization always has a direction that is parallel to the applied electric field, and a magnitude which does not depend on the direction of the electric field is isotropic dielectric material*

***Susceptibility**: The electric susceptibility (χ_e) is a dimensionless proportionality constant that indicates the degree of polarization of a dielectric material in response to an applied electric field. The greater the electric susceptibility, the greater the ability of a material to polarize in response to the field, and thereby reduce the total electric field inside the material (and store energy). It is in this way that the electric susceptibility influences the electric* *permittivity of the material and thus influences many other phenomena in that medium, from the capacitance of capacitors to the speed of light.*

Permittivity: *In electromagnetism, absolute permittivity, often simply called permittivity, usually denoted by the Greek letter ε (epsilon), is the measure of capacitance that is encountered when forming an electric field in a particular medium. More specifically, permittivity describes the amount of charge needed to generate one unit of electric flux in a given medium.*

Dielectric constant*: The dielectric constant (k) of a material is the ratio of its permittivity ε to the permittivity of vacuum* ε_o *, so* $k = \varepsilon / \varepsilon_o$*. The dielectric constant is therefore also known as the relative permittivity of the material.*

Susceptibility, Permittivity and Dielectric constant

The polarization $\vec{P}$ in a homogeneous isotropic dielectric is dependent on the nature of the dielectric; it has the same direction as the resultant electric field and also is dependent on the field. These results are summarized by the equation

$$\vec{P} \propto \vec{E}$$

$$\vec{P} = \chi_e \vec{E}$$

Where χ_e, a scalar quantity, is called the electric susceptibilty of the material. As we know the electric displacement $\vec{D}$ for a dielectric medium is

$$\overrightarrow{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$= \varepsilon_0 \vec{E} + \chi_e \vec{E}$$

$$= (\varepsilon_0 + \chi_e) \vec{E}$$

$$= \varepsilon \vec{E}$$

Where $\varepsilon = \varepsilon_0 + \chi_e$ and is called the permitivity of the material. For air or vacuum P = 0 and so, χ_e = 0, and $\varepsilon = \varepsilon_0$.

The quantities ε and χ_e may depend on the electric field but experimentally it has been found that except for very intense fields they are frequently independent of field. χ_e and ε are, therefore, characteristics of the materials. Materials of this type are called linear dielectric. The electrical behaviour of this material is completely specified by either χ_e or ε. It is, however, more convenient to introduce a dimensionless quantity *K*, called **relative permittivity** or dielectric constant, defined by

$$K = \frac{\varepsilon}{\varepsilon_0} = \frac{\varepsilon_0 + \chi_e}{\varepsilon_0} = 1 + \frac{\chi_e}{\varepsilon_0}$$

In terms of *K*, Gauss' law can be stated as

$$\oint_S \varepsilon_0 K \vec{E} . d\vec{a} = \int_V \rho dV$$

Which is a convenient form, since values of K for different media are listed in tables. The units of $\varepsilon,\ \varepsilon_0\ and\ \chi_e$ are the same, that is, $C^2 N^{-1} m^{-2}$. The units of D (and P) is Cm^{-2}

References:

1. Physics (5^{th} edition) – Halliday, Resnick, Krane

2. Concepts of Electricity and Magnetism – A.K. Rafiqullah, A.K. Roy, M.S. Huq