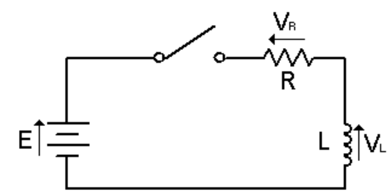
DC and AC Circuits

DC Circuits with LR

1. Growth of current in a circuit containing a resistance and inductance

Consider a circuit having an inductance L and a resistance R connected in series to a cell of steady e.m.f E (Fig.1). When the key K is pressed, there is a gradual growth of current in the circuit from zero to maximum value I_0. Let I be the instantaneous current at any instant.



Then, the induced back e.m.f $\varepsilon = -L\frac{dI}{dt}$

So $$E = RI + L\frac{dI}{dt} \qquad (1)$$

When the current reaches the maximum value I_0, the back e.m.f, $L\frac{dI}{dt} = 0$

Hence $$E = RI_0 \qquad (2)$$

Substituting this value for E in Eq. (1),

$$RI_0 = RI + L\frac{dI}{dt}$$

$$R(I_0 - I) = L\frac{dI}{dt}$$

$$\frac{dI}{(I_0 - I)} = \frac{R}{L}dt$$

Integrating with respect to time we get,

$$\int \frac{dI}{(I_0 - I)} = \int \frac{R}{L}dt$$

$$-\ln(I_0 - I) = \frac{R}{L}t + K \qquad (3)$$

Where K is the constant of integration.

When $t = 0, I = 0$, $\therefore$ $-\ln(I_0) = K$. Substituting this value of K in Eq. (3),

$$-\ln(I_0 - I) = \frac{R}{L}t + -\ln(I_0)$$

$$\ln\left(\frac{I_0 - I}{I_0}\right) = -\frac{R}{L}t$$

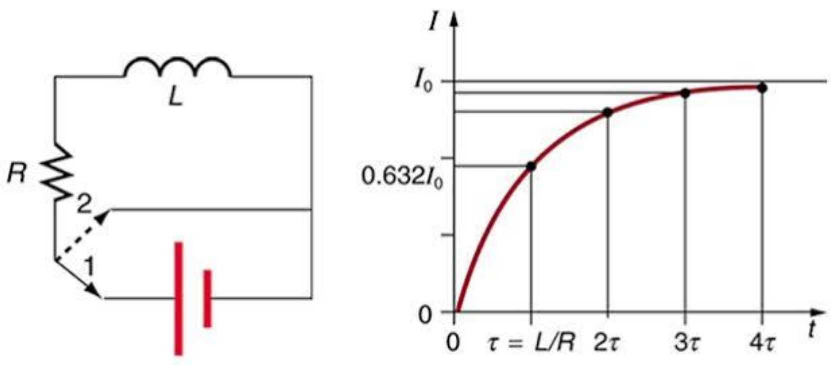
$$\frac{I_0 - I}{I_0} = e^{-\left(\frac{R}{L}\right)t}$$

$$1 - \frac{I}{I_0} = e^{-\left(\frac{R}{L}\right)t}$$

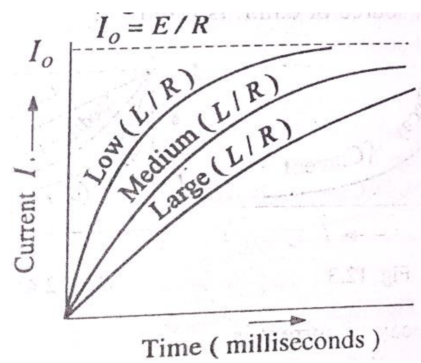
$$I = I_0\left\{1 - e^{-\left(\frac{R}{L}\right)t}\right\}$$

Eq. (4) gives the value of the instantaneous current in the LR circuit. The quantity $\left(\tau = \frac{L}{R}\right)$ is called the time Constant of the circuit. If $\boldsymbol{t = \tau = \frac{L}{R}}$ then $I = I_0\{1 - e^{-1}\} = 0.632\ I_0$

Thus, the time constant L/R of a L-R circuit is the time taken by the current to grow from zero to 0.632 times the steady maximum value of current in the circuit. Greater the value of L/R, longer is the time taken by the current I to reach its maximum value



(a) (b)



2. Decay of Current in a Circuit Containing L and R

When the circuit is broken, an induced e.m.f, equal to $-L\frac{dI}{dt}$ is again produced in the inductance L and it slows down the rate of decay of the current. The current in the circuit decays from maximum value I_0 to zero. During the decay, let I be the current at time t. In this case E = 0. The e.m.f equation (1) for the decay of current is

$$0 = RI + L\frac{dI}{dt}$$

$$RI = -L\frac{dI}{dt}$$

$$\frac{dI}{I} = -\frac{R}{L}dt$$

Integrating with respect to time we get,

$$\int \frac{dI}{I} = \int \frac{R}{L} dt$$

$$-\ln I = \frac{R}{L} t + K$$

Where K is the constant of integration.

When $t = 0, I = 0$, $\therefore$ $-\ln(I_0) = K$. Substituting this value of K

$$-\ln I = \frac{R}{L} t + -\ln(I_0)$$

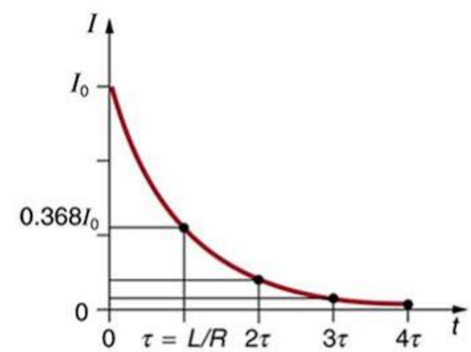
$$\ln\left(\frac{I}{I_0}\right) = -\frac{R}{L} t$$

$$\frac{I}{I_0} = e^{-\left(\frac{R}{L}\right)t}$$

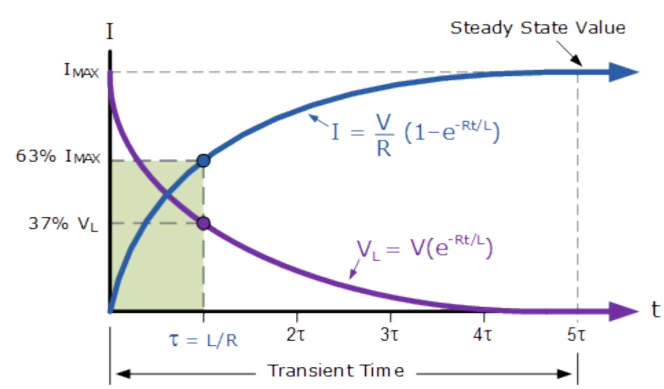
$$\frac{I}{I_0} = e^{-\left(\frac{R}{L}\right)t}$$

$$I = I_0 e^{-\left(\frac{R}{L}\right)t} \qquad (2)$$

Eq, (2) represents the current at any instant t during decay. A graph between current and time is shown in figure. When $\boldsymbol{t = \tau = \frac{L}{R}}$ then $I = I_0 e^{-1} = 0.368\, I_0$. Therefore, the time constant L/R of a R-L circuit may also be defined as the time in which the current in the circuit falls to l/e of its maximum value when external source of e.m.f. is removed.



This graph shows that the growth and decay curves are complementary.



Problem 1. An e.m.f of 10 volt is applied to a circuit having a resistance of 10 ohms and an inductance of 0.5 henry. Find the time required by the current to attain 63.2% of its final value. What is the time constant of the circuit?

Sol. Given $\frac{I}{I_0} = \frac{63.2}{100}$; $R = 10$ ohms ; L= 0.5 henry

We know,

$$I = I_0\left\{1 - e^{-\left(\frac{R}{L}\right)t}\right\}$$

$$\frac{I}{I_0} = 1 - e^{-\left(\frac{R}{L}\right)t}$$

$$e^{-\left(\frac{R}{L}\right)t} = 1 - \frac{I}{I_0}$$

$$\left(\frac{R}{L}\right)t = ln\left(1 - \frac{I}{I_0}\right)$$

$$t = \frac{L}{R} ln\left(1 - \frac{I}{I_0}\right) = \frac{0.5}{10} ln\left(1 - \frac{63.2}{100}\right) = 0.05 \text{ sec}$$

The time constant of the circuit is $\tau = \frac{L}{R} = \frac{0.5}{10} = \frac{1}{20}$ sec

Problem 2: An inductance of 500 mH and a resistance of 5 ohms are connected in series with an e.m.f of 10 volts. Find the final current. If now the cell is removed and the two terminals are connected together, find the current after (i) 0.05 sec. and (ii) 0.2 sec.

Sol. Final current $I_0 = \frac{E}{R} = \frac{10}{5} = 2A$

During discharge, $I = I_0 e^{-\left(\frac{R}{L}\right)t}$

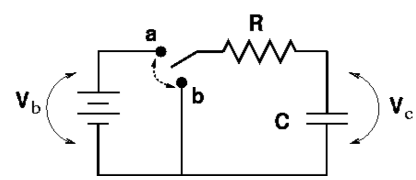
When t = 0.2 sec., $I = 2e^{-\left(\frac{5}{500\times10^{-3}}\right)\times0.2} = 0.271A$

When t = 0.05 sec., $I = 2e^{-\left(\frac{5}{500\times10^{-3}}\right)\times0.05} = 1.213A$

DC Circuits with CR

3. Growth of Charge in a capacitor in CR circuit

A capacitor C and a resistance R are connected to a cell of e.m.f E through a Morse key K. When the key is pressed, a momentary current I flows through R. At any instant t, let Q be the charge on the capacitor of capacitance C.



Potential difference across the capacitor = Q/C

Potential difference across the resistor = RI

The e.m.f equation of the circuit is

$$E = \frac{Q}{C} + RI$$

$$E = \frac{Q}{C} + R\frac{dQ}{dt}$$

Where $I = \frac{dQ}{dt}$. The capacitor continues getting charged till it attains the maximum charge Q_0. At that instant $I = \frac{dQ}{dt} = \frac{dQ_0}{dt} = 0$. The potential difference across the capacitor is $E = \frac{Q_0}{C}$.

i.e., when, $Q = Q_0$, $\frac{dQ}{dt} = \frac{dQ_0}{dt} = 0$ and $E = \frac{Q_0}{C}$

$\therefore$

$$\frac{Q_0}{C} = \frac{Q}{C} + R\frac{dQ}{dt}$$

$$(Q_0 - Q) = CR\frac{dQ}{dt}$$

$$\frac{dQ}{(Q_0 - Q)} = \frac{dt}{CR}$$

Integrating,

$$-\ln(Q_0 - Q) = \frac{t}{CR} + k$$

Where k is a constant. When t = 0, Q = 0. So $-\ln(Q_0) = k$

$$-\ln(Q_0 - Q) = \frac{t}{CR} - \ln(Q_0)$$

$$\ln\left(\frac{Q_0 - Q}{Q_0}\right) = -\frac{t}{CR}$$

$$\frac{Q_0 - Q}{Q_0} = e^{-\left(\frac{t}{CR}\right)}$$

$$1 - \frac{Q}{Q_0} = e^{-\left(\frac{t}{CR}\right)}$$

$$\frac{Q}{Q_0} = 1 - e^{-\left(\frac{t}{CR}\right)}$$

$$Q = Q_0\left\{1 - e^{-\left(\frac{t}{CR}\right)}\right\}$$

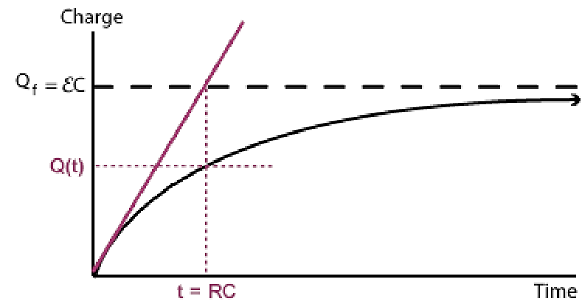
The term $\boldsymbol{\tau = CR}$ is called time constant of the circuit.

At the end of time $t = \tau = CR,\ Q = Q_0\{1 - e^{-1}\}$ = 0.632 Q_0. **Thus, the time constant may be defined as the time taken by the capacitor to get charged to 0.632 times its maximum value**.

The growth of charge is shown in Fig. 12.7. The rate of growth of charge is

$$\frac{dQ}{dt} = \frac{Q_0}{CR} e^{-\left(\frac{t}{CR}\right)}$$

Thus it is seen that smaller the product CR, the more rapidly does the charge grow on the capacitor. The rate of growth of the charge is rapid in the beginning and it becomes less and less as the charge approaches nearer and nearer the steady value.



4. Decay of charge (Discharging of a Capacitor through Resistance)

Let the capacitor having charge Q0 be now discharged by releasing the Morse key K (Fig. 12.6). The charge flows out of the capacitor and this constitutes a current. In this case E = 0.

$$0 = \frac{Q}{C} + RI$$

$$\frac{Q}{C} + R\frac{dQ}{dt} = 0$$

$$\frac{dQ}{Q} = -\frac{1}{CR}dt$$

Integrating,

$$\ln Q = -\frac{t}{CR} + k$$

When t = 0, Q = Q_0. So $\ln(Q_0) = k$

$$\ln Q = -\frac{t}{CR} + \ln(Q_0)$$

$$\ln\left(\frac{Q}{Q_0}\right) = -\frac{t}{CR}$$

$$\frac{Q}{Q_0} = e^{-\left(\frac{t}{CR}\right)}$$

$$Q = Q_0 e^{-\left(\frac{t}{CR}\right)}$$

The term $\boldsymbol{\tau = CR}$ is called time constant of the circuit.

This shows that the charge in the capacitor decays exponentially and becomes zero after infinite interval of time. The rate of discharge is

$$I = \frac{dQ}{dt} = -\frac{Q_0}{CR} e^{-\left(\frac{t}{CR}\right)} = -\frac{Q}{CR}$$

Thus, smaller the time-constant CR, the quicker is the discharge of the capacitor.

In this eqn, if we put t = CR, then $Q = Q_0 e^{-(1)}$ = 0.368 Q_0. Hence time constant may also be defined as the time taken by the current to fall from maximum to 0.368 of its maximum value.

<u>Example:</u> A capacitor of capacitance 0.1 μF is first charged and then discharged through a resistance of 10 $M\Omega$. Find the time, the potential will take to fall to half of its original value.

Solution:

$$Q = Q_0 e^{-\left(\frac{t}{CR}\right)}$$

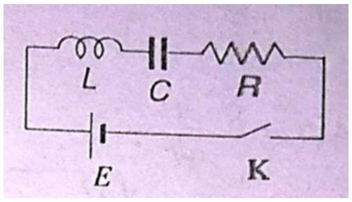
$$\frac{Q}{Q_0} = \frac{CQ}{CQ_0} = \frac{V}{V_0} = e^{-\left(\frac{t}{CR}\right)}$$

$$\ln\left(\frac{V}{V_0}\right) = -\frac{t}{CR}$$

$$t = -RC \ln\left(\frac{V}{V_0}\right) = -10 \times 10^6 \times 0.1 \times 10^{-6} \ln\left(\frac{\frac{1}{2}V_0}{V_0}\right) = 0.6931\, s \qquad \text{Ans.}$$

DC Circuits with LCR in Series

5.Growth of charge in a Circuit with Inductance, Capacitance and Resistance



Consider a circuit containing an inductance L, capacitance C and resistance R joined in series to a cell of e.m.f E. When the key K is pressed, the capacitor is charged. Let Q be the charge on the capacitor and I the current in the circuit at an instant t during charging. Then, the potential difference across the capacitor is Q/C and the self-induced e.m.f in the inductance coil is L(dI/dt), both being opposite to the direction of E. The p.d. - across the resistance R is RI.

The equation of e.m.f is

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E \qquad (1)$$

$$L\frac{d}{dt}\left(\frac{dQ}{dt}\right) + R\frac{dQ}{dt} + \frac{Q}{C} = E$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q - CE}{C} = 0$$

Md. Saifur Rahman, Lecturer, Department of Physics, University of Rajshahi/ICE/Chapter-2/2019

$$\frac{d^2Q}{dt^2}+\frac{R}{L}\frac{dQ}{dt}+\frac{Q-CE}{LC}=0$$

Putting $\frac{R}{L}=2b$ and $\frac{1}{LC}=k^2$, we have

$$\frac{d^2Q}{dt^2}+2b\frac{dQ}{dt}+k^2(Q-CE)=0 \tag{2}$$

Let $x=Q-CE$, then $\frac{d^2x}{dt^2}=\frac{d^2Q}{dt^2}$ and $\frac{dx}{dt}=\frac{dQ}{dt}$ so

$$\frac{d^2x}{dt^2}+2b\frac{dx}{dt}+k^2x=0 \tag{3}$$

Hence the most general solution of Eq. (3) is

$$x=Ae^{[-b+\sqrt{b^2-k^2}]t}+Be^{[-b-\sqrt{b^2-k^2}]t}$$

Where A and B are arbitrary constant. Now CE = Q_0= final steady charge on the capacitor.

$$x=Q-CE=Q-Q_0$$

Hence $$Q-Q_0=Ae^{[-b+\sqrt{b^2-k^2}]t}+Be^{[-b-\sqrt{b^2-k^2}]t}$$

$$Q=Q_0+Ae^{[-b+\sqrt{b^2-k^2}]t}+Be^{[-b-\sqrt{b^2-k^2}]t} \tag{4}$$

Using initial conditions: at t = 0, Q = 0

$$0 = Q_0 + (A + B)$$

$$\text{or } A + B = -Q_0 \tag{5}$$

Differentiating eqn 4

$$\frac{dQ}{dt}=A\left[-b+\sqrt{b^2-k^2}\right]e^{[-b+\sqrt{b^2-k^2}]t}+B\left[-b-\sqrt{b^2-k^2}\right]e^{[-b-\sqrt{b^2-k^2}]t}$$

At t = 0, $\frac{dQ}{dt}=0$, so

$$0=A\left[-b+\sqrt{b^2-k^2}\right]+B\left[-b-\sqrt{b^2-k^2}\right]$$

$$\sqrt{b^2-k^2}[A-B]=b[A+B]=-bQ_0$$

$$A-B=\frac{-bQ_0}{\sqrt{b^2-k^2}} \tag{6}$$

Solving Eqs. (5) and (6),

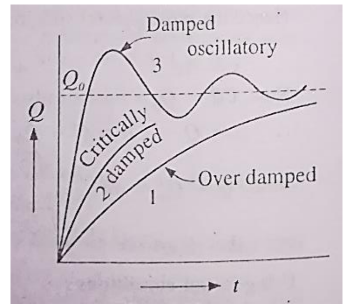
$$A=-\frac{1}{2}Q_0\left(1+\frac{b}{\sqrt{b^2-k^2}}\right) \tag{7}$$

$$B=-\frac{1}{2}Q_0\left(1-\frac{b}{\sqrt{b^2-k^2}}\right) \tag{8}$$

Substituting the values of A and B in Eq. (4), we have

$$Q = Q_0 - \frac{1}{2}Q_0\left(1+\frac{b}{\sqrt{b^2-k^2}}\right)e^{[-b+\sqrt{b^2-k^2}]t} - \frac{1}{2}Q_0\left(1-\frac{b}{\sqrt{b^2-k^2}}\right)e^{[-b-\sqrt{b^2-k^2}]t}$$

$$Q = Q_0 - \frac{1}{2}Q_0e^{-bt}\left[\left(1+\frac{b}{\sqrt{b^2-k^2}}\right)e^{\sqrt{b^2-k^2}t} + \left(1-\frac{b}{\sqrt{b^2-k^2}}\right)e^{-\sqrt{b^2-k^2}t}\right] \tag{9}$$



<u>Case I</u>. If $\boldsymbol{b^2 > k^2}$, $\sqrt{b^2-k^2}$ is real. The charge on the capacitor grows exponentially with time and attains the maximum value Q_0 asymptotically, (curve 1 of Fig). The charge is known as over damped or dead beat.

<u>Case II</u>. If $\boldsymbol{b^2 = k^2}$, the charge rises to the maximum value Q_0 in a short time (curve 2 of Fig.). Such a charge is called critically damped.

<u>Case III</u>. If $\boldsymbol{b^2 < k^2}$, $\sqrt{b^2-k^2}$ is imaginary.

Let $\sqrt{b^2-k^2} = i\omega$ where $i = \sqrt{-1}$ and $\omega = \sqrt{k^2-b^2}$

Eq. (9) may be written as

$$Q = Q_0 - \frac{1}{2}Q_0e^{-bt}\left[\left(1+\frac{b}{i\omega}\right)e^{i\omega t} + \left(1-\frac{b}{i\omega}\right)e^{-i\omega t}\right]$$

$$Q = Q_0 - Q_0e^{-bt}\left[\frac{e^{i\omega t}+e^{-i\omega t}}{2} + \frac{b}{\omega}\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right]$$

$$Q = Q_0 - Q_0e^{-bt}\left[\cos\omega t + \frac{b}{\omega}\sin\omega t\right]$$

$$Q = Q_0\left[1 - \frac{e^{-bt}}{\omega}\{\omega\cos\omega t + b\sin\omega t\}\right]$$

Let $\omega = k\sin\alpha$ and , $b = k\cos\alpha$ so that $\tan\alpha = \frac{\omega}{b}$.

$$Q = Q_0\left[1 - \frac{e^{-bt}}{\omega}\{k\sin\alpha\ \cos\omega t + k\cos\alpha\sin\omega t\}\right]$$

$$Q = Q_0\left[1 - \frac{ke^{-bt}}{\omega}\sin(\omega t+\alpha)\right]$$

$$Q = Q_0\left[1 - \frac{e^{-\frac{R}{2L}t}\sqrt{\frac{1}{LC}}}{\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}\sin\left\{\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\right)t+\alpha\right\}\right]$$

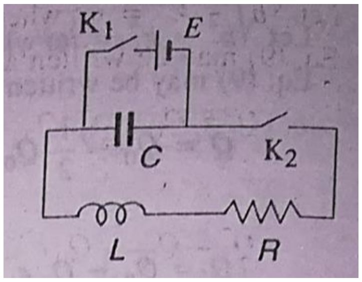
This equation represents a damped oscillatory charge as shown by the curve (3). The charge oscillates above and below Q0 till it finally settles down to Q0 value. The frequency of oscillation in the circuit is given by

$$f = \frac{\omega}{2\pi} = \frac{\sqrt{k^2 - b^2}}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

When R = 0,

$$f = \frac{1}{2\pi\sqrt{LC}}$$

6. Discharge of a Capacitor through an inductor and a Resistor in series (Decay of charge in LCR circuit)



Consider a circuit containing a capacitor of capacitance C, an inductance L and a resistance R are joined in series. E is a cell. K_2 is kept open. The capacitor is 'charged to maximum charge Q_0 by closing the key K_1. On opening K_1 and closing key K_2 the capacitor discharges through the inductance L and resistance R. Let I be the current in the circuit and Q be the charge in the capacitor at any instant during discharge. The circuit equation then is

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = 0$$

$$L\frac{d}{dt}\left(\frac{dQ}{dt}\right) + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0 \qquad (1)$$

Putting $\frac{R}{L} = 2b$ and $\frac{1}{LC} = k^2$, we have

$$\frac{d^2Q}{dt^2} + 2b\frac{dQ}{dt} + k^2Q = 0 \qquad (2)$$

Hence the most general solution of Eq. (2) is

$$Q = Ae^{[-b+\sqrt{b^2-k^2}]t} + Be^{[-b-\sqrt{b^2-k^2}]t} \qquad (3)$$

Where A and B are arbitrary constant.

Using initial conditions: at t = 0, Q = Q_0

$$Q_0 = A + B$$

$$\text{or } A + B = Q_0 \qquad (4)$$

Differentiating eqn.3

(5)

$$\frac{dQ}{dt} = A\left[-b+\sqrt{b^2-k^2}\right]e^{\left[-b+\sqrt{b^2-k^2}\right]t} + B\left[-b-\sqrt{b^2-k^2}\right]e^{\left[-b-\sqrt{b^2-k^2}\right]t}$$

At t = 0, $\frac{dQ}{dt} = 0$, so

$$0 = A\left[-b+\sqrt{b^2-k^2}\right] + B\left[-b-\sqrt{b^2-k^2}\right]$$

$$\sqrt{b^2-k^2}[A-B] = b[A+B] = bQ_0 \tag{6}$$

$$A - B = \frac{bQ_0}{\sqrt{b^2-k^2}}$$

Solving Eqs. (4) and (6),

$$A = \frac{1}{2}Q_0\left(1+\frac{b}{\sqrt{b^2-k^2}}\right) \tag{7}$$

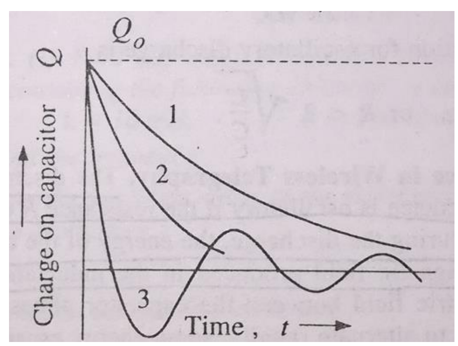
$$B = \frac{1}{2}Q_0\left(1-\frac{b}{\sqrt{b^2-k^2}}\right) \tag{8}$$

Substituting the values of A and B in Eq. (3), we have

$$Q = \frac{1}{2}Q_0\left(1+\frac{b}{\sqrt{b^2-k^2}}\right)e^{\left[-b+\sqrt{b^2-k^2}\right]t} + \frac{1}{2}Q_0\left(1-\frac{b}{\sqrt{b^2-k^2}}\right)e^{\left[-b-\sqrt{b^2-k^2}\right]t}$$

$$Q = \frac{1}{2}Q_0e^{-bt}\left[\left(1+\frac{b}{\sqrt{b^2-k^2}}\right)e^{\sqrt{b^2-k^2}t} + \left(1-\frac{b}{\sqrt{b^2-k^2}}\right)e^{-\sqrt{b^2-k^2}t}\right] \tag{9}$$

Case I. If $\boldsymbol{b^2 > k^2}$, $\sqrt{b^2-k^2}$ is real and positive and the charge of the capacitor decays exponentially, becoming zero asymptotically (curve 1 of Fig). This discharge is known as *over damped, non-oscillatory* or *dead beat*.



Case II. If $\boldsymbol{b^2 = k^2}$, $Q = Q_0(1+bt)e^{-bt}$

This represents a non-oscillatory discharge. This discharge is known as critically damped (Curve 2 of Fig.). The charge decreases to zero exponentially in a short time.

Case III. If $\boldsymbol{b^2 < k^2}$, $\sqrt{b^2-k^2}$ is imaginary.

Let $\sqrt{b^2-k^2} = i\omega$ where $i = \sqrt{-1}$ and $\omega = \sqrt{k^2-b^2}$

Eq. (9) may be written as

$$Q = \frac{1}{2}Q_0e^{-bt}\left[\left(1+\frac{b}{i\omega}\right)e^{i\omega t} + \left(1-\frac{b}{i\omega}\right)e^{-i\omega t}\right]$$

$$Q = Q_0e^{-bt}\left[\frac{e^{i\omega t}+e^{-i\omega t}}{2} + \frac{b}{\omega}\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right]$$

$$Q = Q_0 e^{-bt}\left[\cos\omega t + \frac{b}{\omega}\sin\omega t\right]$$

$$Q = \frac{Q_0 e^{-bt}}{\omega}(\omega\cos\omega t + b\sin\omega t)$$

Let $\omega = k\sin\alpha$ and , $b = k\cos\alpha$ so that $\tan\alpha = \frac{\omega}{b}$.

$$Q = \frac{Q_0 e^{-bt}}{\omega}(k\sin\alpha\ \cos\omega t + k\cos\alpha\sin\omega t)$$

$$Q = \frac{kQ_0 e^{-bt}}{\omega}\sin(\omega t + \alpha)$$

$$Q = Q_0\frac{e^{-\frac{R}{2L}t}\sqrt{\frac{1}{LC}}}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}\sin\left\{\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right)t + \alpha\right\}$$

This equation represents a damped oscillatory charge as shown by the curve (3). The charge oscillates above and below Q_0 till it finally settles down to Q_0 value. The frequency of oscillation in the circuit is given by

$$f = \frac{\omega}{2\pi} = \frac{\sqrt{k^2 - b^2}}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

When R = 0,

$$f = \frac{1}{2\pi\sqrt{LC}}$$

The condition for oscillatory discharge is

$$\frac{1}{LC} > \frac{R^2}{4L^2}$$

$$R < 2\sqrt{\frac{L}{C}}$$

Importance in Wireless Telegraphy

The discharge of a capacitor through an inductance is oscillatory if the resistance R of the circuit is less than $2\sqrt{\frac{L}{C}}$. During the discharge, the energy of the charged capacitor is stored in a magnetic field produced in the inductance coil then again back in the electric field between the capacitor plates, and so on. The fields are caused to alternate rapidly, some energy escapes from the circuit permanently in the form of electro-magnetic waves which travel through space with the speed of light. These waves form the basis of wireless telegraphy. Messages can be transmitted from one place to another with the help of codes.

Example I: If a battery, of e.m.f 100 volts, is connected in series with an inductance of 10 mH, a capacitor of 0.05 μF and a resistance of 100 Ω, find (i) the frequency of the oscillatory current, (ii) the logarithmic decrement and (iii) the final capacitor charge.

Solution: The frequency of oscillation in the circuit is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{10 \times 10^{-3} \times 0.05 \times 10^{-6}} - \frac{100^2}{4 \times (10 \times 10^{-3})^2}} = 7076.6\ Hz$$

The logarithmic decrement $\delta = \frac{RT}{2L} = \frac{R}{2Lf}$

The logarithmic decrement is defined as the natural log of the ratio of the amplitudes of any two successive peaks

$$\delta = \frac{100}{2 \times 10 \times 10^{-3} \times 7076.6}$$

$$= 0.707$$

The final capacitor charge Q_0 = EC = $100 \times 0.05 \times 10^{-6}$ = 5 μC.

Example 2: A charged capacitor of capacitance 0.01 μF is made to discharge through a circuit consisting of a coil of inductance 0.1 henry and an unknown resistance. What should be the maximum value of the unknown resistance, if the discharge of the capacitor is to be oscillatory?

Solution: If R is the maximum value of the resistance for the discharge to be oscillatory, then

$$\frac{1}{LC} = \frac{R^2}{4L^2}$$

$$R = 2\sqrt{\frac{L}{C}} = 2\sqrt{\frac{0.1}{0.01 \times 10^{-6}}} = 6324\ \Omega$$

Example 3: (i) Find out whether the discharge of a capacitor through a circuit containing the following elements, is oscillatory. C = 0.2 μF, L = 10 mH, R = 250 Ω. (ii) If so, find the frequency. (iii) Calculate the maximum value of the resistance possible so as to make the discharge oscillatory.

Solution: (i) The condition for oscillations is that $\frac{1}{LC} < \frac{R^2}{4L^2}$

$$R < 2\sqrt{\frac{L}{C}}$$

We have $2\sqrt{\frac{10\times10^{-3}}{0.2\times10^{-6}}} = 447\ \Omega$

It is given that R = 250 Ω. So $R < 2\sqrt{\frac{L}{C}}$. Therefore the discharge is oscillatory.

(ii) The frequency of oscillations is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{10\times10^{-3}\times0.2\times10^{-6}} - \frac{250^2}{4\times(10\times10^{-3})^2}} = 54\times10^6\ Hz$$

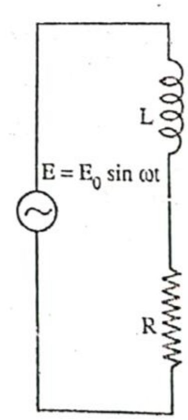
(3) The maximum possible resistance of oscillatory discharge is given by the equation

$$R = 2\sqrt{\frac{L}{C}} = 2\sqrt{\frac{10\times10^{-3}}{0.2\times10^{-6}}} = 447\ \Omega$$

Impedance: In any circuit the ratio of the effective voltage to the effective current is defined as the, impedance Z of the circuit.

AC Circuits with LR

1. A.C. Circuit Containing Inductance and Resistance in Series



Let an alternating e.m.f $E = E_0 e^{j\omega t}$ be applied to a circuit having an inductance L and a non-inductive resistance R in series.

The potential drop across the inductor is $V_L = j\omega LI$

The potential drop across resistor is $V_R = RI$

Here, I is the current at any instant t. So

$$E = j\omega LI + RI$$

Current in the circuit,

$$I = \frac{E}{R + j\omega L}$$

But

$$I = \frac{E}{Z}$$

Impedance of R-L circuit is Z = $R + j\omega L$. So

$$I = \frac{E_0 e^{j\omega t}}{\sqrt{(R^2+\omega^2L^2)}\, e^{j\theta}} \qquad \left(\text{Where } \tan\theta = \frac{\omega L}{R}\right)$$

$$I = \frac{E_0}{\sqrt{(R^2+\omega^2L^2)}} e^{j(\omega t-\theta)}$$

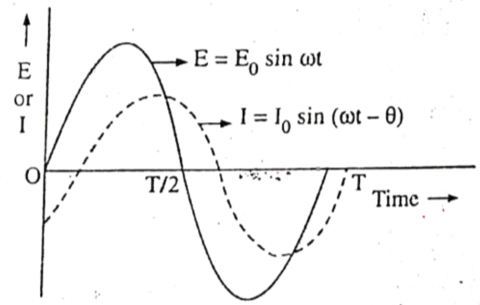
$$I = I_0 e^{j(\omega t-\theta)}$$

Here, $I_0 = \frac{E_0}{\sqrt{(R^2+\omega^2L^2)}}$

It represents the peak value of the current through the circuit. The impedance Z of the circuit is given by the term $\sqrt{(R^2+\omega^2L^2)}$. The current lags in phase behind the e.m.f by an angle

$$\theta = \tan^{-1}\frac{\omega L}{R}$$

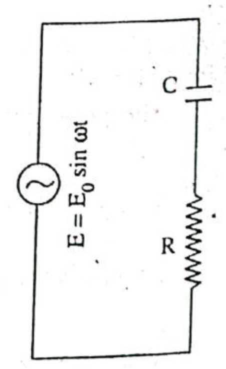
The variation of instantaneous values of e.m.f and current with time are represented graphically in Fig.



AC Circuits with CR

2. A.C. circuit containing a resistance R and a capacitance C in series

Let an alternating e.m.f $E = E_0 e^{j\omega t}$ be applied to a circuit having a capacitance C and a non-inductive resistance R in series.



The potential drop across the capacitor is $V_C = \frac{1}{j\omega C} I$

The potential drop across resistance is $V_R = RI$

Here, I is the current at any instant t. So

$$E = RI + \frac{1}{j\omega C} I$$

Current in the circuit,

$$I = \frac{E}{R + \frac{1}{j\omega C}}$$

But

$$I = \frac{E}{Z}$$

Impedance of R-L circuit is Z = $R + \frac{1}{j\omega C} = R - \frac{j}{\omega C}$. So

$$\left(\text{Where } \tan\theta = \frac{\frac{1}{\omega C}}{R}\right)$$

$$I = \frac{E_0 e^{j\omega t}}{\sqrt{\left(R^2 + \frac{1}{\omega^2 C^2}\right)}\ e^{-j\theta}}$$

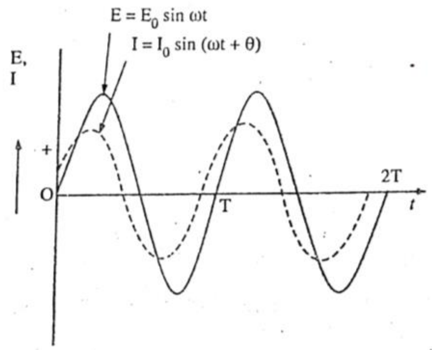
$$I = \frac{E_0}{\sqrt{\left(R^2 + \frac{1}{\omega^2 C^2}\right)}} e^{j(\omega t + \theta)}$$

$$I = I_0 e^{j(\omega t + \theta)}$$

Here, $I_0 = \frac{E_0}{\sqrt{\left(R^2+\frac{1}{\omega^2 C^2}\right)}}$

It represents the peak value of the current through the circuit. The value of the impedance Z of the circuit is given by the term $\sqrt{\left(R^2 + \frac{1}{\omega^2 C^2}\right)}$. The current lags in phase behind the e.m.f by an angle

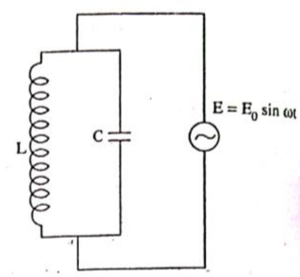
$$\theta = \tan^{-1} \frac{\frac{1}{\omega C}}{R}$$



AC Circuits with LCR

3. Parallel Resonant Circuit

A parallel resonant circuit consisting of an inductance L and a capacitance C connected in parallel to the alternating e.m.f. is shown in Fig. Let the applied, e.m.f be $E = E_0 \sin \omega t$



Current through the inductance lags behind the applied e.m.f by $\frac{\pi}{2}$ and is given by,

$$I_L = \frac{E_0}{\omega L} \sin\left(\omega t - \frac{\pi}{2}\right)$$

Here, ωL is the reactance of the inductor. The current through the capacitor leads the applied emf by $\frac{\pi}{2}$ and is given by

$$I_C = \frac{E_0}{\frac{1}{\omega C}} \sin\left(\omega t + \frac{\pi}{2}\right)$$

Here, $\frac{1}{\omega C}$ is the reactance of the capacitance. The total current through the circuit is given by $I = I_L + I_C$

$$I = \frac{E_0}{\omega L} \sin\left(\omega t - \frac{\pi}{2}\right) + \frac{E_0}{\frac{1}{\omega C}} \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$I = -\frac{E_0}{\omega L}\cos\omega t + \frac{E_0}{\frac{1}{\omega C}}\cos\omega t$$

$$I = E_0\left(\omega C - \frac{1}{\omega L}\right)\cos\omega t$$

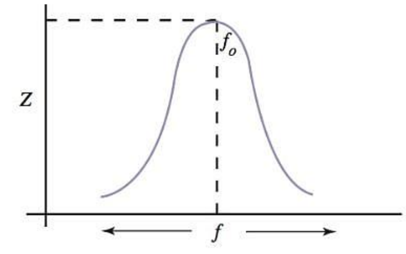
If for a particular frequency f_0, $\omega C = \frac{1}{\omega L}$, then from above Eq., the current becomes zero. The circuit now offers an infinite impedance. Such a circuit which offers an infinite impedance to the A.C. is called a parallel resonant circuit. The frequency is called the resonant frequency.

Thus at f_0, we have

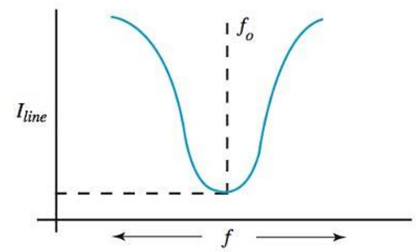
$$\omega C = \frac{1}{\omega L}$$

$$2\pi f_0 C = \frac{1}{2\pi f_0 L}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$



A



B

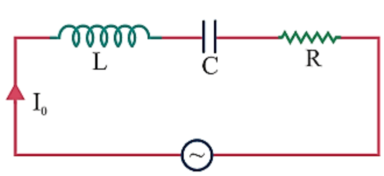
The variation of the current with frequency is shown in Fig.

Rejector Circuit:

The parallel resonant circuit does not allow the current of the same frequency as the natural frequency of the circuit. Thus it can be sued to suppress the current of this particular frequency out of currents of many other frequencies. Hence the circuit is known as a rejector of filter circuit.

4. AC Circuit Containing Resistance, Inductance and Capacitance in series (Series Resonance Circuit)

Let an alternating e.m.f $E = E_0 \sin\omega t$ be applied to a circuit containing a resistance R, inductance L and capacitance C in series. Let at any instant, I be the current in the circuit and Q be the charge on the capacitor.



$$E = E_0 \sin\omega t$$

The potential drop across the resistance = RI

The E.M.F. induced in the inductance = $L\frac{dI}{dt}$

The potential across the plates of the capacitor = $\frac{Q}{C}$. So

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E_0 \sin\omega t$$

Differentiating with respect to t,

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}\frac{dQ}{dt} = E_0\omega\cos\omega t \tag{1}$$

Let the trial solution be of the form $I = I_0\sin(\omega t - \varphi)$ (2)

Where I_0 and φ are constants to be determined.

$$\frac{dI}{dt} = I_0\omega\cos(\omega t - \varphi)$$

$$\frac{d^2I}{dt^2} = -I_0\omega^2\sin(\omega t - \varphi)$$

Substituting these values of I, $\frac{dI}{dt}$ and $\frac{d^2I}{dt^2}$ in Eq. (1), we get

$$-LI_0\omega^2\sin(\omega t - \varphi) + RI_0\omega\cos(\omega t - \varphi) + \frac{1}{C}I_0\sin(\omega t - \varphi) = E_0\omega\cos\omega t$$

$$\left(-L\omega^2 + \frac{1}{C}\right)I_0\sin(\omega t - \varphi) + RI_0\omega\cos(\omega t - \varphi) = E_0\omega\cos\{(\omega t - \varphi) + \varphi\}$$

$$= E_0\omega[\cos(\omega t - \varphi)\cos\varphi - \sin(\omega t - \varphi)\sin\varphi]$$

$$= E_0\omega\cos(\omega t - \varphi)\cos\varphi - E_0\omega\sin(\omega t - \varphi)\sin\varphi$$

Equating the coefficients of $\sin(\omega t - \varphi)$ and $\cos(\omega t - \varphi)$ on either side,

$$\left(-L\omega^2 + \frac{1}{C}\right)I_0 = -E_0\omega\sin\varphi \tag{3}$$

$$RI_0\omega = E_0\omega\cos\varphi \tag{4}$$

Dividing Eq. (3) by Eq. (4), we get

$$\tan\varphi = \frac{L\omega^2 - \frac{1}{C}}{R\omega} = \frac{\omega L - \frac{1}{\omega C}}{R} \tag{5}$$

Squaring and adding Eqs. (3) and (4), we get

$$\left[\left(-L\omega^2 + \frac{1}{C}\right)^2 + R^2\omega^2\right]{I_0}^2 = {E_0}^2\omega^2$$

$${I_0}^2\left[\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2\right] = {E_0}^2$$

$$I_0 = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \tag{6}$$

Substituting the value of I_0 in Eq. (2), we get

(7)

$$I = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin(\omega t - \varphi)$$

Where $\varphi = tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$

Eq. (7) represents the current at any instant.

The quantity $\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$ is the impedance Z of the circuit. Here ωL and $\frac{1}{\omega C}$ respectively represent inductive reactance X_L and capacitive reactance Xc. Thus $Z = \sqrt{R^2 + (X_L - X_C)^2}$

The current lags in phase behind e.m.f. by an angle $\varphi = tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) = tan^{-1}\left(\frac{X_L - X_C}{R}\right)$. The following three cases arise :

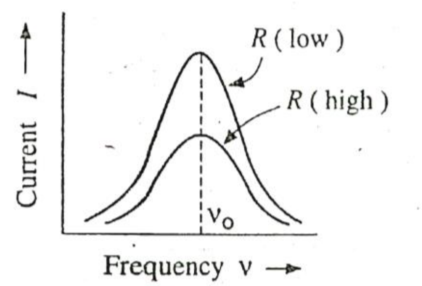
(i) When $X_L > X_C$, φ is positive so that the current lags behind the applied e.m.f.
(ii)When $X_L < X_C$, φ is negative, so that the current leads the applied emf.
(iii) When $X_L = X_C$, φ = 0, and the current is in phase with the e.m.f

N.B: This solution can also be done by J-operator method. Try yourself.

Series Resonant Circuit: The value of current at any instant in a series LCR circuit is given by

$$I = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin(\omega t - \varphi)$$

Where The quantity $Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$ is the impedance of the circuit. At a particular frequency, f_0, $\omega C = \frac{1}{\omega L}$. So that the impedance becomes minimum being given by Z = R. This particular frequency f_0 at which the impedance of the circuit becomes minimum and, therefore the current becomes maximum, is called the ***resonant frequency*** of the circuit. Such a circuit which admits maximum current is called series resonant circuit.



Thus at f_0, we have

$$\omega C = \frac{1}{\omega L}$$

$$2\pi f_0 C = \frac{1}{2\pi f_0 L}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

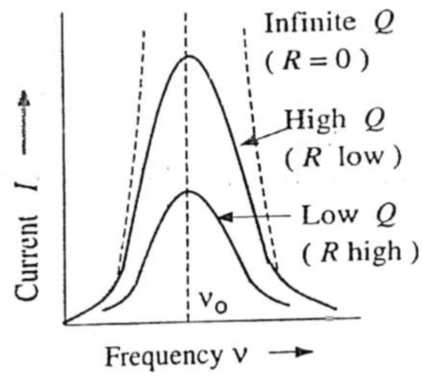
The maximum current in the circuit $I_0 = \frac{E_0}{R}$. The variation of current with frequency of applied voltage is shown in Fig. The sharpness of peak depends upon the resistance R of the circuit. For low resistance, the peak is sharp.

Acceptor Circuit The series resonant circuit is often called an acceptor circuit. By offering minimum impedance to currents at the resonant frequency, it is able to select or accept most readily the current of this one frequency from among those of many frequencies. In radio receivers, the resonant frequency of the circuit is tuned (by changing C) to the frequency of the signal desired to be detected.

The Q-factor

$$\text{Q} - \text{factor} = \frac{\text{Reactance bf the coil at resonance}}{\text{Resistance of the circuit}} = \frac{\omega_0 L}{R}$$

Q-factor determines the degree of selectivity of the circuit while tuning. This is because, for larger values of Q-factor the frequency response curve of the circuit is a steep narrow peak. For smaller values of Q-factor, the frequency response curve is quite flat (Fig.).



Power in AC Circuit Containing Resistance, Inductance and Capacitance

Consider an ac circuit containing resistance, inductance and capacitance. E and I vary continuously with time. Therefore power is calculated at any instant and then its mean is calculated over a complete cycle.

The instantaneous values of the voltage and current are given by

$$E = E_0 \sin \omega t$$

$$I = I_0 \sin(\omega t - \varphi)$$

Where φ is the phase difference between current and voltage. Hence power at any instant is

$$EI = E_0 I_0 \sin \omega t \ \sin(\omega t - \varphi)$$

$$= \frac{1}{2} E_0 I_0 2 \sin \omega t \ \sin(\omega t - \varphi)$$

$$= \frac{1}{2} E_0 I_0 [\cos \varphi - \cos\,(2\omega t - \varphi)]$$

Average power consumed over one complete cycle. is

$$P = \frac{\int_0^T EIdt}{\int_0^T dt}$$

$$= \frac{\int_0^T \frac{1}{2} E_0 I_0 [\cos\varphi - \cos(2\omega t - \varphi)] dt}{T}$$

$$= \frac{1}{2}\frac{E_0 I_0}{T}\left[(\cos\varphi)t - \frac{\sin(2\omega t - \varphi)}{2\omega}\right]_0^T$$

$$= \frac{1}{2}\frac{E_0 I_0}{T}\left[(\cos\varphi)T - 0 - \frac{\sin(2\omega T - \varphi)}{2\omega} + \frac{\sin(-\varphi)}{2\omega}\right]$$

Now $T = \frac{2\pi}{\omega}$ and $\sin(4\pi - \varphi) = \sin(-\varphi)$

$$P = \frac{1}{2}\frac{E_0 I_0 \omega}{2\pi}\left[(\cos\varphi)\frac{2\pi}{\omega} - 0 - \frac{\sin(-\varphi)}{2\omega} + \frac{\sin(-\varphi)}{2\omega}\right]$$

$$P = \frac{1}{2} E_0 I_0 \cos\varphi$$

$$P = \frac{E_0}{\sqrt{2}}\frac{I_0}{\sqrt{2}}\cos\varphi$$

$$P = E_{rms}\, I_{rms} \cos\varphi$$

$$P = E_v\, I_v \cos\varphi$$

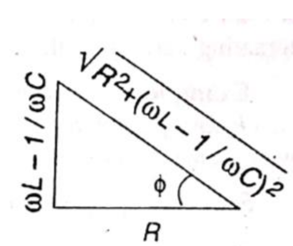
$$\text{(average power)} = \text{(virtual volts)} \times \text{(virtual amperes)} \cos\varphi$$

The term (virtual volts) $\times$ (virtual amperes) is called apparent power and $\cos\varphi$ is called power factor.

$$\text{True power} = (\text{apparent power}) \times \text{power factor}$$

As $\cos\varphi$ is the factor by which the product of voltage and current must be multiplied to give the power dissipated, it is known as the 'power factor' of the circuit. For a circuit containing resistance, capacitance and inductance in series,

$$\tan\varphi = \frac{\omega L - \frac{1}{\omega C}}{R}$$



the expression for the power factor is

$$\cos\varphi = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

Special cases:

(1) In a purely resistive circuit, φ = 0 or cos φ = 1. So, true power = $E_v\ I_v$

(2) In a purely inductive circuit, current lags behind the applied e.m.f by 90° so that 90° or cos φ = 0. Thus true power consumed = 0

(3) In a purely capacitive circuit, current leads the applied voltage by 90° so that φ = -90° or cos (- 90°) = cos 90° = true power = 0

(4) In an ac circuit containing a resistance and inductance in series,

Power factor, $\cos\varphi = \frac{R}{\sqrt{R^2+(\omega L)^2}}$

(5) In an ac circuit containing a capacitance C and a resistance R in series, $\cos\varphi = \frac{R}{\sqrt{R^2+\left(\frac{1}{\omega C}\right)^2}}$

Wattless Current

The average power dissipated during a complete cycle is $E_v\ I_v\ \cos\varphi$. The current in A.C. circuit is said to be wattless when the average power consumed in the circuit is zero. If an AC circuit is purely inductive or purely capacitive with no ohmic resistance, phase angle $\varphi = 90$ so that cos 90 = 0 or the power consumed is zero. The current in such a circuit does not perform any useful work and is rightly called the wattless or idle current. In this situation, the circuit does not consume any power, though it offers a resistance to the flow of alternating current in it. It is the principle of choke coil.

N.B.: Practice mathematical problems from reference books for all chapter.

References:

1. Electricity and magnetism – R. Murugeshan
2. Internet