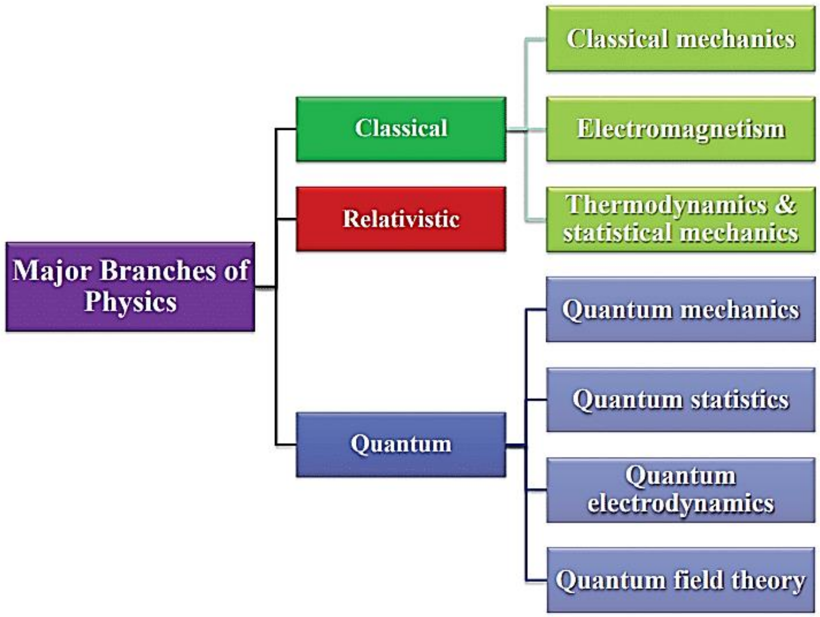
Lagrangian Formulation

Preliminary Lecture

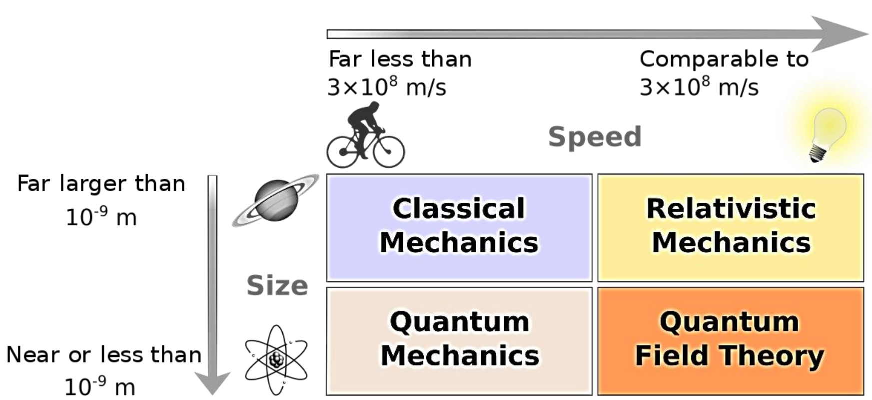


Consideration of size→ **Macroscopic** and **Microscopic**

Star, planet, etc.
which visible to naked eye

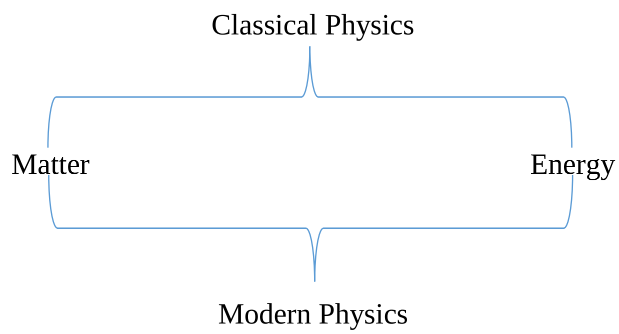
Atoms, molecules, electron etc.
which visible with microscope

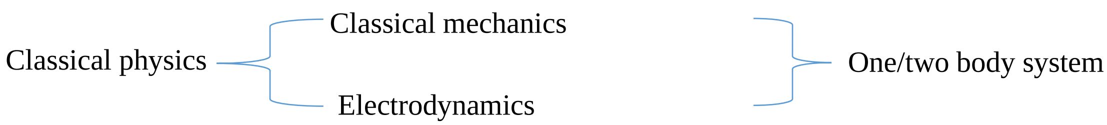
Roadmap in physics:

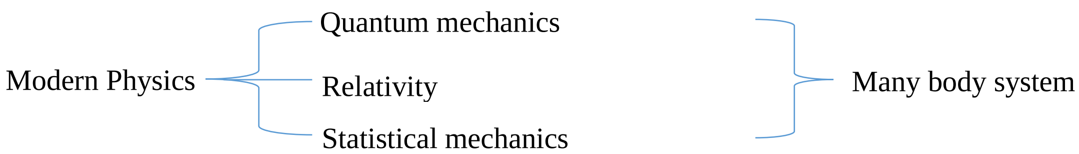


Classical mechanics today has come into wide use to denote that part of mechanics where the objects in question are neither too big, nor too small interacting objects, more precisely, systems of an atomic scale so that a close agreement between theory and experiment is desirable. These extreme cases are dealt within general relativity and quantum mechanics respectively.

- Classical physics → Two distinct aspect of nature
    - Matter: localized
    - Energy: Wave, spread in space







Classical physics refers to theories of **physics** that exist before modern, more complete, or more widely applicable theories. Sir Isaac Newton known as the father of classical physics. **Classical physics** are the physics that were made before the 20^{th} century. This part of physics studies things like movement, light, gravity, and electricity.

Mechanics:

Mechanics (Greek μηχανική) is the area of physics concerned with the motions of macroscopic objects. Forces applied to objects result in displacements, or changes of an object's position relative to its environment. This branch of physics has its origins in Ancient Greece with the writings of Aristotle and Archimedes.

Classical Mechanics:

Classical mechanics describes the motion of macroscopic objects, from projectiles to parts of machinery, and astronomical objects, such as spacecraft, planets, stars and galaxies.

If the present state of an object is known it is possible to predict by the laws of classical mechanics how it will move in the future (determinism) and how it has moved in the past (reversibility).

The earliest development of classical mechanics is often referred to as Newtonian mechanics. It consists of the physical concepts employed and the mathematical methods invented by Isaac Newton, Gottfried Wilhelm Leibniz and others in the 17th century to describe the motion of bodies under the influence of a system of forces.

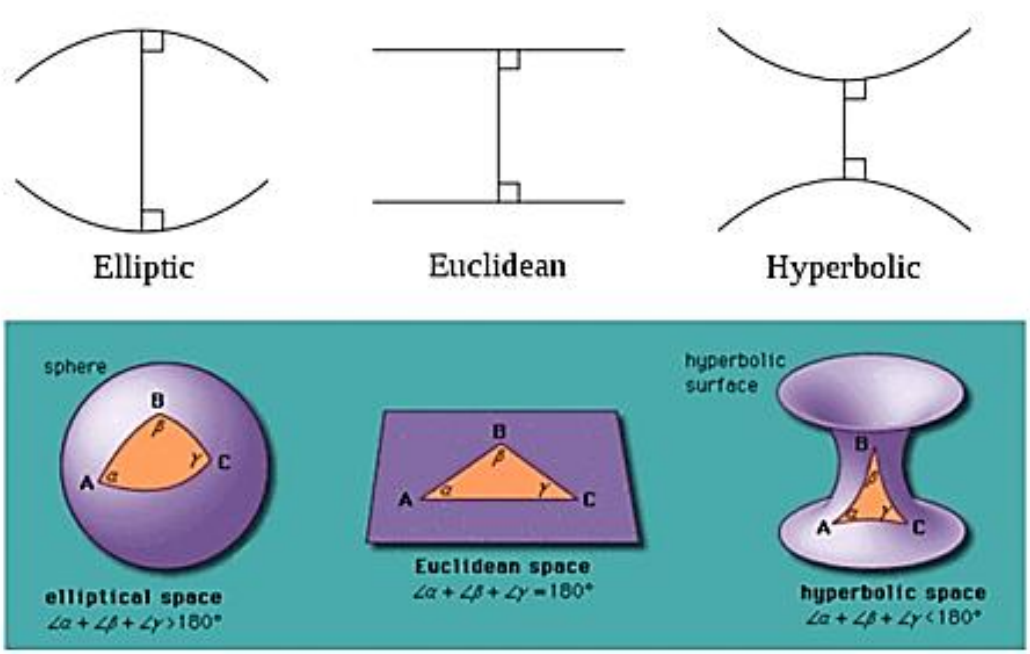
Later, more abstract methods were developed, leading to the reformulations of classical mechanics known as Lagrangian mechanics and Hamiltonian mechanics. These advances, made predominantly in the 18th and 19th centuries, extend substantially beyond Newton's work, particularly through their use of analytical mechanics. They are, with some modification, also used in all areas of modern physics.

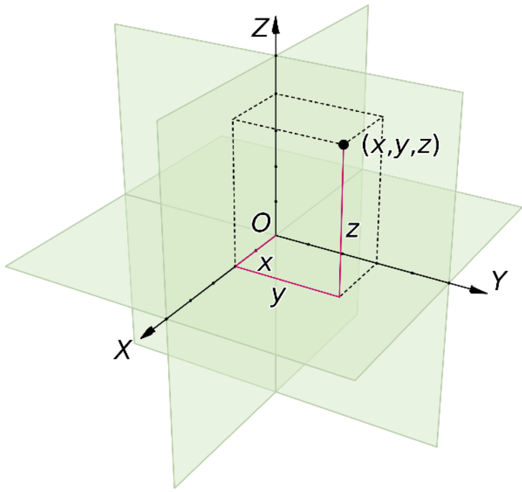
Classical mechanics provides extremely accurate results when studying large objects that are not extremely massive and speeds not approaching the speed of light. When the objects being examined have about the size of an atom diameter, it becomes necessary to introduce the other major sub-field of mechanics: quantum mechanics. To describe velocities that are not small compared to the speed of light, special relativity is needed. In case that objects become extremely massive, general relativity becomes applicable. However, a number of modern sources do include relativistic mechanics into classical physics, which in their view represents classical mechanics in its most developed and accurate form.

Coordinate System

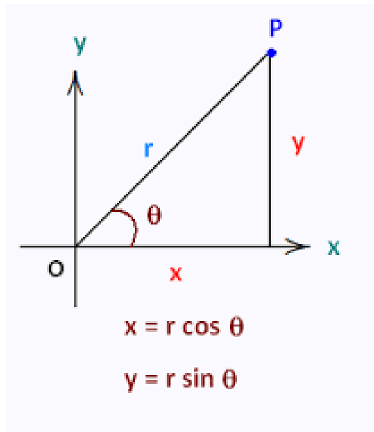
In geometry, a coordinate system is a system that uses one or more numbers, or coordinates, to uniquely determine the position of the points or other geometric elements on a manifold such as Euclidean space.

In geometry, a two- or three-dimensional space in which the axioms and postulates of Euclidean geometry apply is knows as Euclidean space.

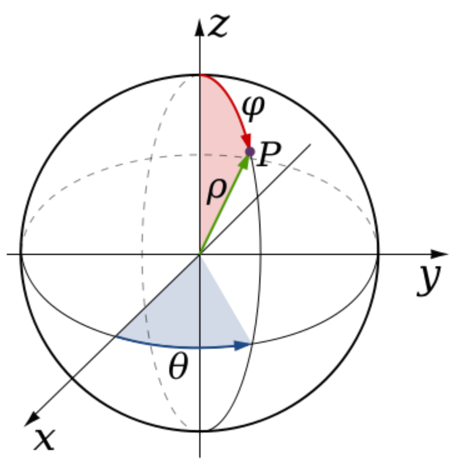




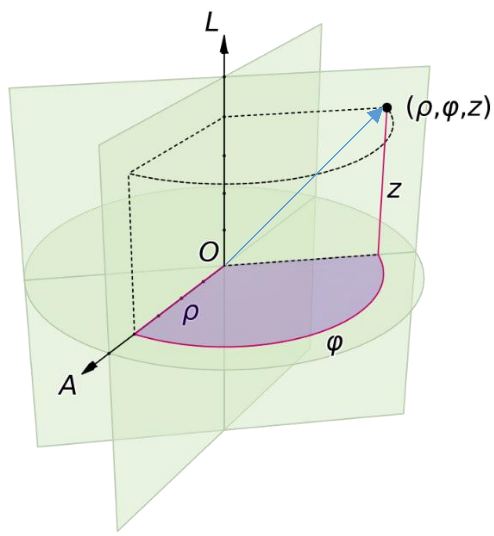
Cartesian coordinate system (x, y, z)



Polar Coordinate System (r, θ)



Spherical coordinate system (ρ, θ, φ)



Cylindrical coordinate system (ρ, φ, z)

Conservative system: A conservative system is a system in which work done by a force is

1. Independent of path.
2. Equal to the difference between the final and initial values of an energy function.
3. Completely reversible.

The two most notable conservative systems are gravitational and electric fields. With gravity for example, the gravitational potential energy acquired or lost by a mass depends only on the difference between heights (or between distances from the origin of the force), and not on the path taken to get from one state to the other.

Contrast a conservative system with a system involving friction in which the work done to get between states *does* depend on the path taken and is not reversible.

Reference:

- **Classical mechanics – Gupta, Kumar, Sharma**
- **Classical mechanics – Goldstein, G**
- **Physics part 1 and part ll – David Halliday and Robert Resnick**
- http://103.79.117.242/ru_profile/public/teacher/21907273/profile#content

Or, Academic→Department→Physics→Faculty member→ Md. Saifur Rahman→Course Materials

- **Internet**

Lagrangian Formulation

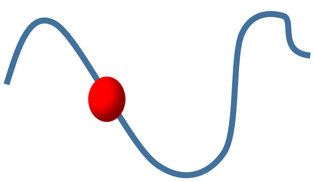
Formulation: *The action of creating or preparing something.*
A material or mixture prepared according to a particular formula.

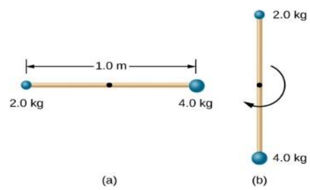
Lagrangian mechanics: This is a reformulation of classical mechanics, introduced by the Italian-French mathematician and astronomer Joseph-Louis Lagrange in 1788.

In Lagrangian mechanics, the trajectory of a system of particles is derived by solving the Lagrange equations in one of two forms: either the Lagrange equations of the first kind, which treat constraints explicitly as extra equations, often using Lagrange multipliers; or the Lagrange equations of the second kind, which incorporate the constraints directly by judicious choice of generalized coordinates. In each case, a mathematical function called the ***Lagrangian*** *is a function of the generalized coordinates, their time derivatives, and time, and contains the information about the dynamics of the system.*

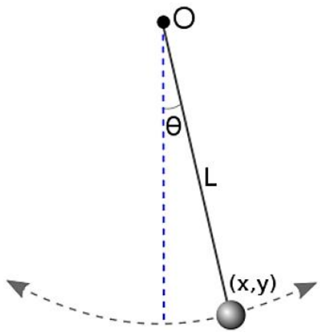
Constraints:

The constraints are the restriction on a body which limit the motion of that body. In classical mechanics, the motion of the bodies is constrained in some way, for example, a massive bead may be constrained to move along a bent wire of certain shape; a massive cylinder may be rolling along a surface (but not sliding or flying around); or two masses may be connected by a rigid stick of fixed length. In each of these cases there are forces acting on the constrained bodies, which limits the motion of that body. These forces are called **constraint force**.





In the above examples, the wire produces a force on the bead, the plane acts by the force of friction on the cylinder, and the stick pulls or pushes on the two masses. These forces may vary in time and we do not know the magnitude of these forces in advance. We know, however, that these forces are at every time exactly such as to guarantee that the constraints hold. The bead would fly away if there were no forces acting on it, but the wire provides a force that keeps the bead in place. The two masses connected by a rigid stick experience a force from the stick that is exactly necessary to keep them at a constant distance from each other. This is what it means that the stick is "rigid".



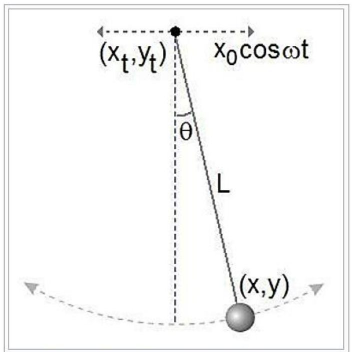
Constraints are classified as:

1. **Scleronomic**: If the equations of constraints do not contain the time as an explicit variable and the equation of constraints can be described by generalized coordinates, then such type of constraints are called **scleronomic** constraints.
   In this case the mechanical system is **scleronomous.**

2. **Rheonomic**: constraint relations depend explicitly on time
   In this case the mechanical system is **rheonomous.**

In the figure, the string is attached at the top end to a pivot and at the bottom end to a weight. Being inextensible, the string has a constant length as there is no time dependency. Therefore, this system is scleronomous; it obeys the scleronomic constraint

$$\sqrt{x^2 + y^2} - L = 0$$



A simple pendulum with oscillating pivot point

The situation changes if the pivot point is moving, e.g. undergoing a simple harmonic motion

$$x_t = x_0 \cos \omega t$$

Where x_0 is the amplitude, ω is the angular frequency, and t is the time.

Although the top end of the string is not fixed, the length of this inextensible string is still a constant. The distance between the top end and the weight must stay the same. Therefore, this system is rheonomous; it obeys the rheonomic constraint

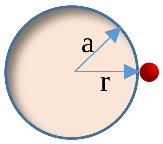
$$\sqrt{(x - x_0 \cos \omega t)^2 + y^2} - L = 0$$

1. **Holonomic**: constraint relations are or can be made independent of velocities ($\dot{x}, \dot{y}, \dot{z}$)
2. **Non-holonomic:** constraint relations are not holonomic (depend on velocities)

Holonomic constraints are relations between the position variables (and possibly time) which can be expressed in the following form:

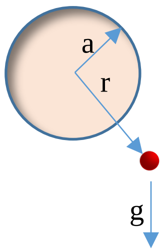
$$f(q_1, q_2, q_3, \ldots, q_n, t) = 0$$

Where {q_1, q_2, q_3… q_n} are the n coordinates which describe the system. For example, the motion of a particle constrained to lie on the surface of a sphere is subject to a holonomic constraint, but if the particle is able to fall off the sphere under the influence of gravity, the constraint becomes non-holonomic. For the first case the holonomic constraint may be given by the equation:



$$r^2 - a^2 = 0$$

Where r is the distance from the center of a sphere of radius a. Whereas the second non-holonomic case may be given by:



$$r^2 - a^2 \geq 0$$

Velocity-dependent constraints such as:

$$f(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t) = 0$$

are not usually holonomic. The constraints involved in the motion of the molecules in a gas container are non-holonomic. An object rolling on a rough surface without slipping involves non-holonomic constraint in the description of its motion.

Note that there are some special cases of velocity-dependent constraints which can actually be integrated to give holonomic constraints, these are holonomic-in-disguise. For example, consider a general velocity-dependent constraint:

$$A\dot{x} + B = 0$$

This doesn't look holonomic, and in general it's not. However, in the case where there exists a function f such that A=$\frac{\partial f}{\partial x}$ and B=$\frac{\partial f}{\partial t}$, then

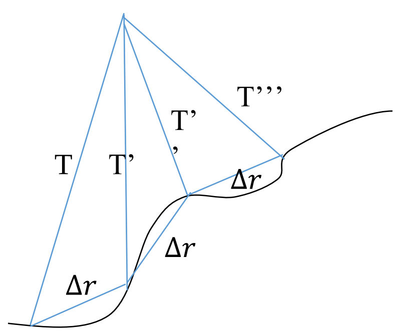
$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial t} = 0$$

$$df = 0$$

Which means we may integrate this to yield a holonomic constraint f=0 (constant)

1. **Conservative:** In this case, total mechanical energy of the system is conserved while performing the constrained motion. Constraint forces do not do any work.
2. **Dissipative**: constraint forces do work and total mechanical energy is not conserved.

Usually the constraint forces act in a direction perpendicular to the surface of constraints at every point on it, while the motion of the object is parallel to the surface at every point. In such cases the work done by constraint forces is zero. One obvious exception is, of course, the frictional force due to sliding which does work for real displacements. Another exception is the rheonomic constraint for which the constraint force need not act perpendicular to the real displacement. We see from the figure that



1. **Bilateral**: at any point on the constraint surface both the forward and backward motions are possible. Constraint relations are not in the form of inequalities but are in the form of equation.
2. **Unilateral**: at some points no forward motion is possible. Constraint relations are expressed in the form of inequalities.

If one slowly pushes a box around on a horizontal surface z=0 (and it is a reasonable assumption that the box will not lose contact with the surface because of downward gravity), then one can replace the actual unilateral constraint z≥0 with an effective description in terms of a bilateral constraint z=0.
The constraint function $f(q)=q^2$ is a bilateral constrain. Here may be q=0 and negative and positive, hence bilateral.

Some typical examples constraints are as follows:

- *The bob of a pendulum must remain a fixed distance from the point of support.*
- *The particles of a rigid body must maintain fixed distances from each other.*
- *A particle sliding on a wire must not leave the wire.*
- *The contact particle of a body rolling on a fixed surface must be at rest.*

The rolling condition is a kinematical constraint since it involves the velocity of a particle. All the other constraints are geometrical

Degrees of Freedom:

The number of independent ways in which a mechanical system can move without violating any constraint which may be imposed such that its configuration changes, is called the number of degrees of freedom of the system. It is indicated by the least possible number of coordinates to describe the system completely.

For example, when a single particle moves in space, it has three degrees of freedom, but if it is constrained to move along a certain space curve, it has only one. Similarly, a rigid body rotating about an axis fixed in space has only one degree of freedom—that of rotation angle about the axis. We then conclude that imposing constraints is a way of simplifying the problems mathematically in that the number of equations of motion are reduced to the same number as the number of degrees of freedom. In a system of N particles subject to *k* independent constraints expressible in *k* equations of the form

$$g_1 (x_1, y_1, z_1 \ldots\ldots z_N, t) = a_1$$

$$g_2 (x_1, y_1, z_1 \ldots\ldots z_N, t) = a_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$g_k (x_1, y_1, z_1 \ldots\ldots z_N, t) = a_k \qquad (1)$$

Then the number of degrees of freedom $f = 3N-k$

In the above equations, g_1, g_2, g_k are k specified functions of 3N coordinates and possibly of time if constraints depend on time explicitly.

Initially there are 360 ***joints*** *in the* ***human*** *body. Degrees of freedom of human hand is 7 whereas the total human body possess 244 degree of freedom. The number of joints are 230 in matured human body. And, many of the joints have one degree of freedom and some have more than one DoF.*

Generalized Coordinates:

To describe the configuration of a system, we select the smallest possible number of variables. These are called the generalized coordinates of the system. Generalized coordinates refer to the parameters that describe the configuration of the system relative to some reference configuration. These parameters must uniquely define the configuration of the system relative to the reference configuration.

We shall not restrict our choice only to Cartesian coordinates. In many cases these are not the most convenient coordinates in terms of which we are to describe the motion of the system. A set of generalized coordinates is any set of coordinates which describe the configuration.

A system of N particles, free from constraints has 3N independent coordinates. If there exist holonomic constraints expressed in (k) equations in the form $f(r_1, r_2, \ldots t)$ then we may use to eliminate k of 3N coordinates. The number of independent coordinates will be given by

$$f = 3N - k \tag{2}$$

The number of independent coordinates in D dimensions, the original configuration would need ND coordinates, and the reduction by constraints means $n = ND - k$. It is ideal to use the minimum number of coordinates needed to define the configuration of the entire system, while taking advantage of the constraints on the system. These quantities are known as generalized coordinates.

This elimination of the independent coordinates can be expressed by the introduction of new 3N-k independent variables $q_1, q_2, q_3 \ldots q_{3N-k}$ in terms of which the old coordinates $r_1, r_2, r_3 \ldots r_N$ can be expressed by the equation of the form

$$r_1 = r_1(q_1, q_2, q_3 \ldots q_{3N-k}, t)$$

$$r_2 = r_2(q_1, q_2, q_3 \ldots q_{3N-k}, t)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$r_N = r_N(q_1, q_2, q_3 \ldots q_{3N-k}, t) \tag{3}$$

The above set of equation can be solved in terms of $q_1, q_2, q_3 \ldots q_{3N-k}$ as

$$q_1 = q_1(r_1, r_2, r_3 \ldots r_N, t)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$q_f = q_f(r_1, r_2, r_3 \ldots r_N, t) \tag{4}$$

This set of new coordinates $q_1, q_2, q_3 \ldots q_{3N-k,}$ is known as generalized coordinates.

In general, we can always express generalized coordinates as some functions of Cartesian coordinates, and possibly function of time as

$$q_1 = q_1(x_1, y_1, z_1; x_2, y_2, z_2; \ldots x_N, y_N, z_N; t)$$

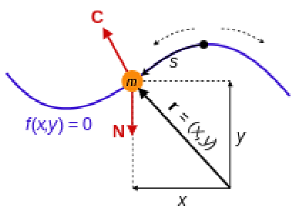
$$q_2 = q_2(x_1, y_1, z_1; x_2, y_2, z_2; \ldots x_N, y_N, z_N; t)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

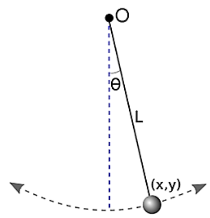
$$q_{3N} = q_{3N}(x_1, y_1, z_1, x_2, y_2, z_2, \ldots x_N, y_N, z_N, t)$$

Above equations are the transformation equation from a set of 3N Cartesian coordinates to 3N generalized coordinates.

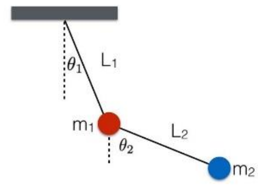
Generalized coordinates are paired with generalized momenta to provide **canonical coordinates** on phase space. Some examples of generalized coordinate:



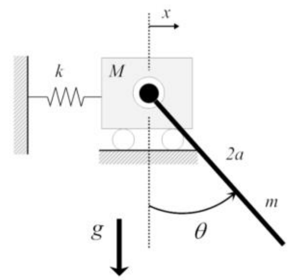
Arc length s



Angle θ



Angle θ_1, θ_2



Displacement x, angle θ

Generalized Displacement

Let us consider a small displacement of an N-particle system defined by changes δr_i in Cartesian coordinates r_i $(i = 1, 2, ..., N)$ with time t held fixed. For the sake of simplicity we may consider an arbitrary virtual displacement δr_i, then since r_i are functions of generalized coordinates defined by the equation for unconstrained system as:

$$\boldsymbol{r}_i = \boldsymbol{r}_i (q_1, q_2, q_3 .....q_{3N}, t) \tag{4}$$

We have from Euler's theorem

$$\delta \boldsymbol{r}_i = \frac{\partial \boldsymbol{r}_i}{\partial q_1} dq_1 + \frac{\partial \boldsymbol{r}_i}{\partial q_2} dq_2 + ... + \frac{\partial \boldsymbol{r}_i}{\partial t} dt$$

$$\delta \boldsymbol{r}_i = \sum_{j=1}^{3N} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j \qquad \text{(as } \delta t = 0\text{)} \tag{5}$$

Euler's theorem

$X = X (x_1, x_2, x_3)$

$$dX = \frac{\partial X}{\partial x_1} dx_1 + \frac{\partial X}{\partial x_2} dx_2 + \frac{\partial X}{\partial x_3} dx_3 \tag{6}$$

We have chosen r_i, to represent the 3N coordinates $(x_1, y_1, ...z_N)$ for notational convenience; each r_i is equivalent to three component coordinates x_1, y_1, z_l and so on. δq_j are called the generalized displacements or virtual arbitrary displacements. If q_j is an angle coordinate, δq_j is an angular displacement.

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Generalized Velocity

Generalized velocity may be described in terms of time derivative of the generalized coordinate q_j, i.e. $\dot{q}_j$, which is then called generalized velocity associated with a particular coordinate q_j. We have for an unconstrained system:

$$\boldsymbol{r}_i = \boldsymbol{r}_i\,(q_1, q_2, q_3.....q_{3N,}\, t)$$

Then

$$\frac{d\boldsymbol{r}_i}{dt} = \sum_{j=1}^{3N} \frac{\partial \boldsymbol{r}_i}{\partial q_j}\frac{\partial q_j}{\partial t} + \sum_{j=1}^{3N} \frac{\partial \boldsymbol{r}_i}{\partial t}\frac{\partial t}{\partial t}$$

$$\dot{\boldsymbol{r}}_i = \sum_{j=1}^{3N} \frac{\partial \boldsymbol{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \boldsymbol{r}_i}{\partial t} \qquad (7)$$

If the N-system contains k constraints, the number of generalized coordinates is $3N - k = f$ and in that case

$$\dot{\boldsymbol{r}}_i = \sum_{j=1}^{f} \frac{\partial \boldsymbol{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \boldsymbol{r}_i}{\partial t} \qquad (8)$$

Note that if the generalized coordinate q_j involves both Cartesian and angle coordinates, the generalized velocity associated with a Cartesian coordinate x is just the corresponding linear velocity while generalized velocity with an angular coordinate θ is the corresponding angular velocity. If a generalized coordinate has the dimensions of momentum, the generalized velocity will have the dimensions of force and so on.

Generalized Acceleration

Components of accelerations are given by differentiating eqn. of velocities. Here the generalized velocity is

$$\dot{\boldsymbol{r}}_i = \sum_{j=1}^{3N} \frac{\partial \boldsymbol{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \boldsymbol{r}_i}{\partial t} \qquad (8)$$

Differentiating it again w.r.t. time, we get

$$\ddot{\boldsymbol{r}}_i = \frac{d}{dt}\left(\sum_{j=1}^{3N} \frac{\partial \boldsymbol{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \boldsymbol{r}_i}{\partial t}\right)$$

$$= \sum_{j=1}^{3N} \frac{d}{dt}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\dot{q}_j\right) + \frac{d}{dt}\left(\frac{\partial \boldsymbol{r}_i}{\partial t}\right)$$

$$= \sum_{j=1}^{3N} \dot{q}_j \frac{d}{dt}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\right) + \sum_{j=1}^{3N}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\right)\ddot{q}_j + \frac{\partial \dot{\boldsymbol{r}}_i}{\partial t}$$

$$= \sum_{j=1}^{3N} \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j}\dot{q}_j + \sum_{j=1}^{3N}\frac{\partial \boldsymbol{r}_i}{\partial q_j}\ddot{q}_j + \frac{\partial \dot{\boldsymbol{r}}_i}{\partial t} \tag{9}$$

Putting for $\dot{r}_i$ from eq. (8), on changing index j to k, we get

$$\ddot{\boldsymbol{r}}_i = \sum_{j=1}^{3N} \frac{\partial}{\partial q_j}\left(\sum_{k=1}^{3N}\frac{\partial \boldsymbol{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \boldsymbol{r}_i}{\partial t}\right)\dot{q}_j + \sum_{j=1}^{3N}\frac{\partial \boldsymbol{r}_i}{\partial q_j}\ddot{q}_j + \frac{\partial}{\partial t}\left(\sum_{k=1}^{3N}\frac{\partial \boldsymbol{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \boldsymbol{r}_i}{\partial t}\right)$$

$$= \sum_{j=1}^{3N}\sum_{k=1}^{3N}\frac{\partial^2 \boldsymbol{r}_i}{\partial q_j \partial q_k}\dot{q}_k\dot{q}_j + \sum_{j=1}^{3N}\frac{\partial^2 \boldsymbol{r}_i}{\partial q_j \partial t}\dot{q}_j + \sum_{j=1}^{3N}\frac{\partial \boldsymbol{r}_i}{\partial q_j}\ddot{q}_j + \sum_{k=1}^{3N}\frac{\partial^2 \boldsymbol{r}_i}{\partial t \partial q_k}\dot{q}_k + \frac{\partial^2 \boldsymbol{r}_i}{\partial t^2} \tag{10}$$

The Cartesian components are not linear functions of components of generalized accelerations $\ddot{q}_j$ alone, but depend quadratically and linearly on the generalized velocity components $\dot{q}_j$ as well. However, in the new approach devised by Lagrange, the computation of second derivatives of generalized coordinates is not required.

Generalized Kinetic Energy

Let us first write down an expression for kinetic energy in terms of generalized velocities. The kinetic energy T of a system of N free particles in terms of Cartesian coordinates is

$$T = \sum_{i=1}^{N}\frac{1}{2}m_i\dot{r}_i^2$$

$$= \sum_{i=1}^{N}\frac{1}{2}m_i(\dot{r}_i.\dot{r}_i) \tag{11}$$

Substituting for $\dot{r}_i$ from equation (8), we obtain

$$T = \sum_{i=1}^{N}\frac{1}{2}m_i\left[\sum_{j=1}^{3N}\frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}\right]\left[\sum_{k=1}^{3N}\frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t}\right]$$

$$= \sum_{i=1}^{N}\frac{1}{2}m_i\left[\sum_{j=1}^{3N}\sum_{k=1}^{3N}\frac{\partial r_i}{\partial q_k}\dot{q}_k\frac{\partial r_i}{\partial q_j}\dot{q}_j + \sum_{j=1}^{3N}\frac{\partial r_i}{\partial q_j}\dot{q}_j\frac{\partial r_i}{\partial t} + \sum_{k=1}^{3N}\frac{\partial r_i}{\partial t}\frac{\partial r_i}{\partial q_k}\dot{q}_k + \left(\frac{\partial r_i}{\partial t}\right)^2\right]$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{3N}\sum_{k=1}^{3N} m_i \frac{\partial r_i}{\partial q_k}\frac{\partial r_i}{\partial q_j}\dot{q}_k\dot{q}_j + \frac{1}{2}\sum_{j=1}^{3N}\sum_{i=1}^{N} m_i \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial t}\dot{q}_j + \frac{1}{2}\sum_{k=1}^{3N}\sum_{i=1}^{N} m_i \frac{\partial r_i}{\partial t}\frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{1}{2}\sum_{i=1}^{N} m_i \left(\frac{\partial r_i}{\partial t}\right)^2$$

Second term on the right side consists of two sums which are identical so that

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{3N}\sum_{k=1}^{3N} m_i \frac{\partial r_i}{\partial q_k}\frac{\partial r_i}{\partial q_j}\dot{q}_k\dot{q}_j + \sum_{k=1}^{3N}\sum_{i=1}^{N} m_i \frac{\partial r_i}{\partial t}\frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{1}{2}\sum_{i=1}^{N} m_i \left(\frac{\partial r_i}{\partial t}\right)^2 \qquad (12)$$

$$= T^{(2)} + T^{(1)} + T^{(0)}$$

Thus, the general kinetic energy in terms of generalized velocities comprises three distinct terms; $T^{(2)}$ contains terms quadratic in generalized velocities and this fact is indicated by a superscripts (2) on T; $T^{(1)}$ containing linear terms and $T^{(0)}$ is independent of generalized velocities.

Generalized Momentum

From eq. (11) we observe that linear momentum, associated with the linear velocity $\dot{x}_i$ is $m_i\dot{x}_i$, is given by

$$P_{x_i} = \frac{\partial T}{\partial \dot{x}_i} = m_i\dot{x}_i \qquad \left[T = \sum_{i=1}^{N}\frac{1}{2}m_i\dot{r}_i^2\right]$$

Then momentum associated with generalized coordinate q_k is similarly defined and is called the generalized momentum p_k associated with a coordinate q_k

$$p_k = \frac{\partial T}{\partial \dot{q}_k} \qquad (13)$$

p_k need not always have dimension (MLT^{-1}) of linear momentum; for, if q_k happens to be an angular coordinate, p_k is the corresponding angular momentum (ML^2T^{-1}). Differentiating eq. (12) w.r.t, q_k we get

$$p_k = \sum_{i=1}^{N}\sum_{j=1}^{3N}\frac{\partial T}{\partial q_k}$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{3N} m_i \frac{\partial r_i}{\partial q_k}\frac{\partial r_i}{\partial q_j}\dot{q}_j + \sum_{i=1}^{N} m_i \frac{\partial r_i}{\partial t}\frac{\partial r_i}{\partial q_k} \qquad (14)$$

Last term will again be absent if the generalized system is stationary. It is again a linear function of generalized velocities.

Problem: The kinetic energy of a particle is given by $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 sin^2\theta\dot{\varphi}^2)$. Calculate the generalized momenta.

Solution: We know, the radial component of generalized momentum is

$$P_r = \frac{\partial T}{\partial \dot{r}} = \frac{\partial}{\partial \dot{r}}\left\{\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 sin^2\theta\dot{\varphi}^2)\right\}$$

$$= \frac{1}{2}m2\dot{r} = m\dot{r}$$

the angular polar component of generalized momentum is

$$P_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left\{\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 sin^2\theta\dot{\varphi}^2)\right\}$$

$$= \frac{1}{2}mr^2 2\dot{\theta} = mr^2\dot{\theta}$$

And the azimuthal component of generalized momentum is

$$P_\varphi = \frac{\partial T}{\partial \dot{\varphi}} = \frac{\partial}{\partial \dot{\varphi}}\left\{\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 sin^2\theta.\dot{\varphi}^2)\right\}$$

$$= \frac{1}{2}mr^2 sin^2\theta.2\dot{\varphi} = mr^2\dot{\varphi}\, sin^2\theta$$

Generalized Force

The definition of generalized force associated with a generalized displacement is given as follows: let us consider the amount of work done by a force $\sum \delta \boldsymbol{F}_i$, on the system during an arbitrary small displacement $\sum \delta \boldsymbol{r}_i$, of the system

$$\delta W = \sum_{i=1}^{N} \boldsymbol{F_i} . \boldsymbol{\delta r_i}$$

$$= \sum_{i=1}^{N} \boldsymbol{F_i} . \sum_{j=1}^{3N} \frac{\partial \boldsymbol{r_i}}{\partial q_j} \delta q_j$$

$$= \sum_{j=1}^{N} Q_j\, \delta q_j \qquad (15)$$

Where

$$Q_j = \sum_{i=1}^{3N} \boldsymbol{F_i} . \frac{\partial \boldsymbol{r_i}}{\partial q_j}$$

We note that Q_j depends on the force acting on the particles and on the coordinate q_j and possibly on time t. It is natural to call Q_j, the generalized force associated with a coordinated q_j. Product of Qj with the arbitrary displacement, that a generalized coordinate suffers, is equal to the work done corresponding to that dis-placement. From eq. (15) it follows that whatever dimension a generalized coordinate has, the product of the generalized force and generalized displacement (coordinate) must have the dimensions of work. For that reason, the generalized force need not always have the dimensions of force.

Generalized Potential

If the forces acting on the system are derivable from a scalar potential V depending on the position only, (i.e. conservative system) then V is the potential energy of the system and we have for work done by the force on the system in an arbitrary displacement δr_i of the system as

$$\delta W = -\delta V$$

(When you do conservative work on an object, the work you do is equal to the negative change in potential energy W=−ΔU. As an example, if you lift an object against Earth's gravity, the work will be −mgh. Gravity is doing work on the object by pulling it towards the Earth, but since you are pushing it in the other direction, the field does negative work when you increase a particle's potential energy.)

$$= -\sum_{i=1}^{N}\left[\frac{\partial V}{\partial x_i}\delta x_i + \frac{\partial V}{\partial y_i}\delta y_i + \frac{\partial V}{\partial z_i}\delta z_i\right] \tag{16}$$

Also

$$\delta W = -\sum_{k=1}^{3N}\frac{\partial V}{\partial q_k}\delta q_k$$

$$= \sum_{k=1}^{3N} Q_k \delta q_k$$

Thus

$$Q_k = -\frac{\partial V}{\partial q_k} \tag{17}$$

In this sense, the definition of Q_k as generalized force is a natural one. Also eq. (17) directly follows from eq. (16): for,

$$\frac{\partial V}{\partial q_k} = \sum_{i=1}^{N}\left[\frac{\partial V}{\partial x_i}\frac{\delta x_i}{\partial q_k} + \frac{\partial V}{\partial y_i}\frac{\delta y_i}{\partial q_k} + \frac{\partial V}{\partial z_i}\frac{\delta z_i}{\partial q_k}\right]$$

$$= \sum_{i=1}^{N}\boldsymbol{F_i}\cdot\frac{\partial \boldsymbol{r_i}}{\partial q_k} = -Q_k$$

When the system is not conservative (i.e., say depends on the generalized velocities $\dot{q}$) we define the generalized force associated with a coordinate q_j as

$$Q_k = -\frac{\partial V}{\partial q_k} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

Where U may be called a 'velocity dependent potential' or 'generalized potential' since it gives rise to generalized force Q_j. The obvious advantage that we have in formulating laws of mechanics in terms of generalized coordinates and the associated mechanical quantities is that the equation of motion then look simpler and can be solved independently of each other since generalized coordinates are all independent—constraints have no effect on them. Equations of motion are ***Lagranges Equation of Motion.***

Virtual Displacement:

Let us consider two possible configurations of a system of particles at a particular instant which are consistent with the forces of constraints. In order to transform from one configuration to the other we need only a small displacement $\delta \boldsymbol{r}_{\boldsymbol{n}}$ of n-th particle from the old to the new position. $\delta \boldsymbol{r}_{\boldsymbol{n}}$ is called the virtual displacement. In the case of virtual displacement, there are finite number of paths in the configuration space. The duration of time in which the force acting on the particle is zero.

Whereas the path which is actually followed by the system in this configuration space with respect to time is referred to as the actual displacement of this system.

D'Alembert's Principle

D'Alembert's form of the principle of virtual work states that a system of rigid bodies is in dynamic equilibrium when the virtual work of the sum of the applied forces and the inertial forces is zero for any virtual displacement of the system.

Statement: "A system of particles moves in such a way that the total virtual work done is equals to zero. i.e.

$$\sum_{i=1}^{N} \left(\vec{\boldsymbol{F}}_i^a - \dot{\vec{\boldsymbol{P}}}_i \right) . \delta \vec{\boldsymbol{r}}_i = o$$

Or it can be said as, "At any given instant, the force applied to a particle are balanced by the inertial force"

Proof:

Let us consider the system is in equilibrium, i.e. total force $\mathbf{F}_i$ on every particle is zero. i.e.

$$\mathbf{F}_i = 0 \tag{1}$$

Then the work done by this force in a small virtual displacement $\delta \boldsymbol{r}_i$ is

$$\sum_i \mathbf{F}_{\boldsymbol{i}} . \delta \boldsymbol{r}_i = 0 \tag{2}$$

Let the total force be expressed as sum of applied force $\mathbf{F}_i^a$ and forces of constraint $\mathbf{f}_i$ i.e.

$$\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{f}_i \tag{3}$$

Then from eqn. 2 and 3 we get,

$$\sum_i \mathbf{F}_i . \delta \boldsymbol{r}_i = \sum_i (\mathbf{F}_i^a + \mathbf{f}_i). \delta \boldsymbol{r}_i = 0 \tag{4}$$

$$\sum_i \mathbf{F}_i^a . \delta \boldsymbol{r}_i + \sum_i \mathbf{f}_i . \delta \boldsymbol{r}_i = 0$$

We now consider the system for which the virtual work of the forces of constraints is zero. An example of such system is, if we consider that, a particle be constrained to move on a smooth surface so that the forces of constraints being perpendicular to the surface while virtual displacement tangential to it, then the virtual work done by forces of constraints will be zero. Thus,

$$\sum_i \mathbf{F}_i^a . \delta \boldsymbol{r}_i = 0 \qquad \text{[Virtual work]} \tag{5}$$

Where, $\sum_i \mathbf{f}_i . \delta \boldsymbol{r}_i = 0$

The equation is termed as principle of virtual work.

To interpret the equilibrium of the system, D'Alembert's conceived that a system will remain in equilibrium under the action of a force equal to the actual force $\mathbf{F}_i$ plus reversed effective force $\dot{\boldsymbol{P}}_i$ i.e. the total force equal to the reverse effective force. Thus,

$$\mathbf{F}_i + \left(-\dot{\boldsymbol{P}}_i\right) = \mathbf{0} \tag{6}$$

$$\mathbf{F}_i - \dot{\boldsymbol{P}}_i = \mathbf{0}$$

$$\mathbf{F}_i - \dot{\boldsymbol{P}}_i = \mathbf{0}$$

So

$$\sum_i (\mathbf{F}_i - \dot{\boldsymbol{P}}_i) . \delta \boldsymbol{r}_i = 0$$

$$\sum_i (\mathbf{F}_i^a + \mathbf{f}_i - \dot{\boldsymbol{P}}_i) . \delta \boldsymbol{r}_i = 0$$

$$\sum_i (\mathbf{F}_i^a - \dot{\boldsymbol{P}}_i) . \delta \boldsymbol{r}_i + \sum_i \mathbf{f}_i . \delta \boldsymbol{r}_i = 0$$

$$\sum_i (\mathbf{F}_i^a - \dot{\boldsymbol{P}}_i) . \delta \boldsymbol{r}_i = 0 \qquad \left[as \ \sum_i \mathbf{f}_i . \delta \boldsymbol{r}_i = \mathbf{0}\right]$$

Since force of constraints are no more in picture, it is better to drop superscript a (as $\mathbf{f}_i = \boldsymbol{o}, \ \mathbf{F}_i = \mathbf{F}_i^a$ **)**. Thus,

$$\sum_i (\mathbf{F}_i - \dot{\boldsymbol{P}}_i) . \delta \boldsymbol{r}_i = 0 \tag{7}$$

This equation is D'Alembert's principle.

(F = ma; in D'Alembert's form, the force F plus the negative of the mass m times acceleration a of the body is equal to zero: F - ma = 0. In other words, the body is in equilibrium under the action of the real force F and the fictitious force -ma. The fictitious force is also called an inertial force and a reversed effective force.)

Lagrange's Equation from D'Alembert's Principle:

The coordinate transformation equations are

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2, q_3 \ldots\ldots q_n, t) \tag{1}$$

So that

$$\frac{d\boldsymbol{r_i}}{dt} = \frac{\partial \boldsymbol{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \boldsymbol{r_i}}{\partial q_2}\frac{dq_2}{dt} + \ldots + \frac{\partial \boldsymbol{r_i}}{\partial t}\frac{dt}{dt}$$

$$\boldsymbol{v_i} = \dot{\boldsymbol{r}}_{\boldsymbol{i}} = \sum_{j=1}^{3N} \frac{\partial \boldsymbol{r_i}}{\partial q_j}\dot{q}_j + \frac{\partial \boldsymbol{r_i}}{\partial t} \tag{2}$$

The infinitesimal displacement δr_i can be connected with δq_j as,

$$\delta \boldsymbol{r_i} = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r_i}}{\partial q_j}\delta q_j + \frac{\partial \boldsymbol{r}_i}{\partial t}\delta t$$

But last term is zero since in virtual displacement only coordinate displacement is considered and not that of time. Therefore,

$$\delta \boldsymbol{r_i} = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r_i}}{\partial q_j}\delta q_j \tag{3}$$

From D'Alembert's principle we know

$$\sum_i (\mathbf{F}_{\boldsymbol{i}} - \dot{\boldsymbol{P}}_{\boldsymbol{i}}).\,\delta \boldsymbol{r_i} = 0$$

Using equation 3 in this equation we get

$$\sum_i (\mathbf{F}_{\boldsymbol{i}} - \dot{\boldsymbol{P}}_{\boldsymbol{i}}).\sum_{j=1}^{n} \frac{\partial \boldsymbol{r_i}}{\partial q_j}\delta q_j = 0$$

$$\sum_{ij} \mathbf{F}_{\boldsymbol{i}}\cdot\frac{\partial \boldsymbol{r_i}}{\partial q_j}\delta q_j - \sum_{ij} \dot{\boldsymbol{P}}_{\boldsymbol{i}}\cdot\frac{\partial \boldsymbol{r_i}}{\partial q_j}\delta q_j = 0$$

We know the generalized force is

$$Q_j = \sum_{i=1}^{n} \boldsymbol{F}_i\cdot\frac{\partial \boldsymbol{r_i}}{\partial q_j}$$

$Q_{\boldsymbol{j}}$ have the dimension of force and the product $Q_{\boldsymbol{j}}\delta q_j$ must have the dimension of work. Thus, the above equation takes the form

(4)

$$\sum_{j} Q_j \delta q_j - \sum_{ij} \dot{\boldsymbol{P}}_{\boldsymbol{i}} \cdot \frac{\partial \boldsymbol{r}_{\boldsymbol{i}}}{\partial q_j} \delta q_j = 0$$

Let us evaluate the second term of eqn. 4

$$\sum_{ij} \dot{\boldsymbol{P}}_{\boldsymbol{i}} \cdot \frac{\partial \boldsymbol{r}_{\boldsymbol{i}}}{\partial q_j} \delta q_j = \sum_{ij} m_i \ddot{\boldsymbol{r}}_{\boldsymbol{i}} \cdot \frac{\partial \boldsymbol{r}_{\boldsymbol{i}}}{\partial q_j} \delta q_j$$

$$= \sum_{ij} \left\{ \frac{d}{dt}\left(m_i \dot{\boldsymbol{r}}_{\boldsymbol{i}} \cdot \frac{\partial \boldsymbol{r}_{\boldsymbol{i}}}{\partial q_j} \right) - m_i \dot{\boldsymbol{r}}_{\boldsymbol{i}} \cdot \frac{d}{dt}\left(\frac{\partial \boldsymbol{r}_{\boldsymbol{i}}}{\partial q_j} \right) \right\} \delta q_j$$

$$= \sum_{ij} \left\{ \frac{d}{dt}\left(m_i \boldsymbol{v}_{\boldsymbol{i}} \cdot \frac{\partial \boldsymbol{r}_{\boldsymbol{i}}}{\partial q_j} \right) - m_i \boldsymbol{v}_{\boldsymbol{i}} \cdot \frac{d}{dt}\left(\frac{\partial \boldsymbol{r}_{\boldsymbol{i}}}{\partial q_j} \right) \right\} \delta q_j \qquad (5)$$

Further

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{r}_{\boldsymbol{i}}}{\partial q_j} \right) = \frac{\partial}{\partial q_j}\left(\frac{d\boldsymbol{r}_{\boldsymbol{i}}}{dt} \right) = \frac{\partial \boldsymbol{v}_{\boldsymbol{i}}}{\partial q_j} \qquad (6)$$

Also differentiating eqn. (2) with respect to $\dot{q}_j$, we get

$$\frac{\partial \boldsymbol{v}_{\boldsymbol{i}}}{\partial \dot{q}_j} = \frac{\partial \boldsymbol{r}_{\boldsymbol{i}}}{\partial q_j} \qquad (7)$$

Putting eqn. (6) and (7) in eqn. (5), we get,

$$\sum_{ij} \dot{\boldsymbol{P}}_{\boldsymbol{i}} \cdot \frac{\partial \boldsymbol{r}_{\boldsymbol{i}}}{\partial q_j} \delta q_j = \sum_{ij} \left\{ \frac{d}{dt}\left(m_i \boldsymbol{v}_{\boldsymbol{i}} \cdot \frac{\partial \boldsymbol{v}_{\boldsymbol{i}}}{\partial \dot{q}_j} \right) - m_i \boldsymbol{v}_{\boldsymbol{i}} \cdot \frac{\partial \boldsymbol{v}_{\boldsymbol{i}}}{\partial q_j} \right\} \delta q_j$$

$$= \sum_{j} \left\{ \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\sum_{i} \frac{1}{2} m_i {v_i}^2 \right) \right) - \frac{\partial}{\partial q_j}\left(\sum_{i} \frac{1}{2} m_i {v_i}^2 \right) \right\} \delta q_j$$

$$= \sum_{j} \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right\} \delta q_j$$

Where for $\sum_i \frac{1}{2} m_i {v_i}^2$ is the kinetic energy and can be denoted as T.

With substitution eqn. 4 becomes,

$$\sum_{j} Q_j \delta q_j - \sum_{j} \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right\} \delta q_j = 0$$

Further

$$\sum_{j} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0$$

Since constraints are holonomic, q_j are independent of each other and hence to satisfy above equation the coefficient of each δq_j should separately vanish, i.e.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \tag{8}$$

Case I: Conservative System:

Now we fit this problem to **conservative system** so that above set of equations may be identified as Lagrange's equations of motion. For a conservative system, forces $\mathbf{F}_i$ are derivable from potential function V, the latter being purely dependent on coordinates, i.e.,

$$\boldsymbol{F_i} = -\boldsymbol{\nabla_i} V$$

$$= -\frac{\partial V}{\partial \mathbf{r}_i}$$

$$\boldsymbol{\nabla_i} V = \frac{\partial v\boldsymbol{i}}{\partial x} + \frac{\partial v\boldsymbol{j}}{\partial y} + \frac{\partial v\boldsymbol{k}}{\partial z}$$

$$= \frac{\partial v\boldsymbol{i}.\boldsymbol{i}}{\partial x\boldsymbol{i}} + \frac{\partial v\boldsymbol{j}.\boldsymbol{j}}{\partial y\boldsymbol{j}} + \frac{\partial v\boldsymbol{k}.\boldsymbol{k}}{\partial z\boldsymbol{k}}$$

$$= \frac{\partial v}{\partial \boldsymbol{x}} + \frac{\partial v}{\partial \boldsymbol{y}} + \frac{\partial v}{\partial \boldsymbol{z}}$$

Then generalized force at once can be expressed as

$$Q_j = \sum_{i=1}^{n} \boldsymbol{F_i}.\frac{\partial \boldsymbol{r_i}}{\partial q_j} = -\sum_{i=1}^{n} \boldsymbol{\nabla_i} V.\frac{\partial \boldsymbol{r_i}}{\partial q_j}$$

$$= -\sum_{i=1}^{n} \frac{\partial V}{\partial \mathbf{r}_i}.\frac{\partial \boldsymbol{r_i}}{\partial q_j}$$

$$= -\frac{\partial V}{\partial q_j}$$

$$= \sum_{i=1}^{n} \frac{\partial V}{\partial \mathrm{r}_i}\hat{\mathrm{r}}_i.\frac{\partial r_i}{\partial q_j}\hat{\mathrm{r}}_i$$

$$= \sum_{i=1}^{n} \frac{\partial V}{\partial \mathbf{r}_i}.\frac{\partial \boldsymbol{r_i}}{\partial q_j}$$

Now from eqn. (8)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

Since V is not a function of $\dot{q}_j$.

Recognizing (T-V) as L, the Lagrangian for the conservative system, we see that equations become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

$$\sum_{j=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right] = 0 \tag{9}$$

Which are known as Lagrange's equations of motion for conservative system.

Case II. Non-conservative System:

If potentials are velocity dependent, called generalized potentials, then though the system is not conservative, and the generalized force $Q_j(q_j, \dot{q}_j)$ can be written as

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \tag{10}$$

Putting this value of Q; in equation (8), we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

$$\frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_j}\right) - \frac{\partial (T-U)}{\partial q_j} = 0$$

If we take Lagrangian L = T —U, where U is generalized potential, then above equation is

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right] = 0$$

which is exactly of the same form as eq. (9). An example of of such a type will follow in the calculation of Lagrangian for the case of electromagnetic forces on moving charges (art. applications of Lagrange's equations).

MAXWELL'S EQUATIONS

Name or Description	SI	Gaussian
Faraday's law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$
Ampere's law	$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$	$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{J}$
Poisson equation	$\nabla \cdot \mathbf{D} = \rho$	$\nabla \cdot \mathbf{D} = 4\pi\rho$
[Absence of magnetic monopoles]	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
Lorentz force on charge q	$q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$	$q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$
Constitutive relations	$\mathbf{D} = \epsilon\mathbf{E}$ $\mathbf{B} = \mu\mathbf{H}$	$\mathbf{D} = \epsilon\mathbf{E}$ $\mathbf{B} = \mu\mathbf{H}$

In a plasma, $\mu \approx \mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$ (Gaussian units: $\mu \approx 1$). The permittivity satisfies $\epsilon \approx \epsilon_0 = 8.8542 \times 10^{-12}\,\mathrm{F\,m^{-1}}$ (Gaussian: $\epsilon \approx 1$) provided that all charge is regarded as free. Using the drift approximation $\mathbf{v}_\perp = \mathbf{E} \times \mathbf{B}/B^2$ to calculate polarization charge density gives rise to a dielectric constant $K \equiv \epsilon/\epsilon_0 = 1 + 36\pi \times 10^9 \rho/B^2$ (SI) $= 1 + 4\pi\rho c^2/B^2$ (Gaussian), where ρ is the mass density.

Conversion tables

[edit]

Table 1: Mechanical units

Symbol	Property	SI Unit	Factor	cgs
l	Length	meter (m)	100	centimeter(cm)
m	Mass	kilogram (kg)	1000	gram (g)
t	Time	second (s)	1	second (s)
a	Acceleration	m/s^2	100	galileo (Gal)
F	Force	newton (N)	10^5	dyne (dyn)
W	Energy	joule (J)	10^7	erg (erg)
P	Power	watt (W)	10^7	erg/s

Example: 1 J = 10^7 erg

Table 2: Electric units

Symbol	Property	SI Unit	Factor	Gaussian
I	Electric current	ampere (A)	$10c$	statampere (statA)
Q	Charge	coulomb (C)	$10c$	statcoulomb (statC)
V	Electric potential	volt (V)	$10^6/c$	statvolt (statV)
R	Resistance	ohm (Ω)	$10^5/c^2$	statohm (statΩ)
G	Conductance	siemens (S)	$10^{-5}c^2$	statsiemens (statS)
L	Self-inductance	henry (H)	$10^5/c^2$	abhenry (abH)
C	Capacitance	farad (F)	$10^{-5}c^2$	cm
E	Electric field	V/m	$10^4/c$	statV/cm
ρ	Electric charge density	C/m^3	$c/10^5$	$statC/cm^3$
D	Electric displacement	C/m^2	$4\pi 10^{-3}c$	statV/cm

c is the speed of light in m/s (≈ $3{\cdot}10^8$ m/s).
Example: 1 A = 10*c* statA.

Table 3: Magnetic units

Symbol	Property	Gaussian → SI
Φ	magnetic flux	1 Mx → 10^{-8} Wb = 10^{-8} V·s
B	magnetic flux density	1 G → 10^{-4} T = 10^{-4} Wb/m^2
	magnetic induction	
H	magnetic field	1 Oe → $10^3/(4\pi)$ A/m
m	magnetic moment	1 erg/G = 1 emu → 10^{-3} $A{\cdot}m^2$ = 10^{-3} J/T
M	magnetization	1 erg/(G·cm^3) = 1 emu/cm^3 → 10^3 A/m
4π*M*	magnetization	1 G → $10^3/(4\pi)$ A/m
σ	mass magnetization	1 erg/(G·g) = 1 emu/g → 1 $A{\cdot}m^2$/kg
	specific magnetization	
j	magnetic dipole moment	1 erg/G = 1 emu → 4π · 10^{-10} Wb·m
J	magnetic polarization	1 erg/(G·cm^3) = 1 emu/cm^3 → 4π · 10^{-4} T
χ, κ	susceptibility	1 → 4π
χ_ρ	mass susceptibility	1 cm^3/g → 4π · 10^{-3} m^3/kg
μ	permeability	1 → 4π · 10^{-7} H/m = 4π · 10^{-7} Wb/(A·m)
μ_r	relative permeability	μ → μ_r
w, W	energy density	1 erg/cm^3 → 10^{-1} J/m^3
N, D	demagnetizing factor	1 → 1/(4π)

Mx = maxwell, G = gauss, Oe = oersted ; Wb = weber, V = volt, s = second, T = tesla, m = meter, A = ampere, J = joule, kg = kilogram, H = henry

Scalar potential

Scalar potential, *simply stated, describes the situation where the difference in the potential energies of an object in two different positions depends only on the positions, not upon the path taken by the object in traveling from one position to the other. It is a scalar field in three-space: a directionless value (scalar) that depends only on its location. A familiar example is potential energy due to gravity.*

A scalar potential is a fundamental concept in vector analysis and physics (the adjective scalar is frequently omitted if there is no danger of confusion with vector potential). The scalar potential is an example of a scalar field. Given a vector field $\boldsymbol{F}$, *the scalar potential P is defined such that:*

$$\mathbf{F} = -\boldsymbol{\nabla}P = -\left(\frac{\partial P\boldsymbol{i}}{\partial x} + \frac{\partial P\boldsymbol{j}}{\partial y} + \frac{\partial P\boldsymbol{k}}{\partial z}\right)$$

Vector potential

In vector calculus, a vector potential is a vector field whose curl is a given vector field. This is analogous to a scalar potential, which is a scalar field whose gradient is a given vector field. Formally, given a vector field v, a vector potential is a vector field A such that

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

The vector **A** is called the *magnetic vector potential*.

B is a divergence free field s $\boldsymbol{\nabla}.\boldsymbol{B} = 0$. So $\boldsymbol{\nabla}.(\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$

$F = qVB\,sin\theta \quad [B] = \frac{[MLT^{-2}]}{[IT][LT^{-1}]} = [MT^{-2}I^{-1}]$

Dimension of **B** is $MT^{-2}I^{-1}$.

Dimensions of the Magnetic potential is $MLT^{-1}Q^{-1}$ or $MLT^{-2}I^{-1}$

Lorentz force:

The force F acting on a particle of electric charge q with instantaneous velocity v, due to an external electric field E and magnetic field B, is given by (in SI units):

$$\boldsymbol{F} = q[\boldsymbol{E} + (\boldsymbol{V} \times \boldsymbol{B})]$$

In cgs-Gaussian units, which are somewhat more common among theoretical physicists as well as condensed matter experimentalists, one has instead

Where

$$q_{cgs} = \frac{q_{si}}{\sqrt{4\pi\varepsilon_0}} \qquad E_{cgs} = \sqrt{4\pi\varepsilon_0}\, E_{si} \qquad B_{cgs} = \sqrt{\frac{4\pi}{\mu_0}}\, B_{si} \qquad C = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

$$\boldsymbol{F} = q_{cgs}\sqrt{4\pi\varepsilon_0}\left[\frac{\boldsymbol{E}_{cgs}}{\sqrt{4\pi\varepsilon_0}} + \left(\boldsymbol{V}\times\boldsymbol{B}_{cgs}\right)\sqrt{\frac{\mu_0}{4\pi}}\right]$$

$$\boldsymbol{F} = q_{cgs}\left[\boldsymbol{E}_{cgs} + \frac{1}{C}\left(\boldsymbol{V}\times\boldsymbol{B}_{cgs}\right)\right]$$

Lagrangian for a Charged Particles in an Electromagnetic Field:

In an electromagnetic field the force on a particle of charge, q, is given by

$$\boldsymbol{F} = q\left[\boldsymbol{E} + \frac{1}{C}(\boldsymbol{V}\times\boldsymbol{B})\right]$$

But to incorporate it in Lagrangian formulation, we must express **F** in terms of vector and scalar potentials **A** and ϕ. Thus writing

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$$

and

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{C}\frac{\partial\boldsymbol{A}}{\partial t}$$

Equation for F becomes

$$\boldsymbol{F} = q\left[-\boldsymbol{\nabla}\phi - \frac{1}{C}\frac{\partial\boldsymbol{A}}{\partial t} + \frac{1}{C}(\boldsymbol{V}\times\boldsymbol{\nabla}\times\boldsymbol{A})\right] \tag{1}$$

To write it in a more convenient form, let us write the x-component of all terms on right hand side, i.e.

$$(\boldsymbol{\nabla}\phi)_x = \frac{\partial\phi}{\partial x}$$

and

$$\boldsymbol{V}\times\boldsymbol{\nabla}\times\boldsymbol{A} = \boldsymbol{V}\times\begin{vmatrix}\mathbf{i} & \boldsymbol{j} & \boldsymbol{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}$$

$$= \boldsymbol{V} \times \left\{ \boldsymbol{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \right\}$$

$$= \begin{vmatrix} \mathbf{i} & \boldsymbol{j} & \boldsymbol{k} \\ V_x & V_y & V_z \\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) & \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) & \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \end{vmatrix}$$

$$(\boldsymbol{V} \times \boldsymbol{\nabla} \times \boldsymbol{A})_x = V_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - V_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

$$= V_y\frac{\partial A_y}{\partial x} - V_y\frac{\partial A_x}{\partial y} - V_z\frac{\partial A_x}{\partial z} + V_z\frac{\partial A_z}{\partial x}$$

$$= V_x\frac{\partial A_x}{\partial x} - V_x\frac{\partial A_x}{\partial x} + V_y\frac{\partial A_y}{\partial x} - V_y\frac{\partial A_x}{\partial y} - V_z\frac{\partial A_x}{\partial z} + V_z\frac{\partial A_z}{\partial x}$$

on adding and subtracting the term $V_x\frac{\partial A_x}{\partial x}$

$$(\boldsymbol{V} \times \boldsymbol{\nabla} \times \boldsymbol{A})_x = V_x\frac{\partial A_x}{\partial x} + V_y\frac{\partial A_y}{\partial x} + V_z\frac{\partial A_z}{\partial x} - V_x\frac{\partial A_x}{\partial x} - V_y\frac{\partial A_x}{\partial y} - V_z\frac{\partial A_x}{\partial z}$$

$$= \frac{\partial}{\partial x}(\boldsymbol{V}.\boldsymbol{A}) - \left[V_x\frac{\partial A_x}{\partial x} + V_y\frac{\partial A_x}{\partial y} + V_z\frac{\partial A_x}{\partial z}\right] \tag{2}$$

Further, total time derivative of A_x is

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial y}\frac{dy}{dt} + \frac{\partial A_x}{\partial z}\frac{dz}{dt} + \frac{\partial A_x}{\partial t}$$

$$= \frac{\partial A_x}{\partial x}V_x + \frac{\partial A_x}{\partial y}V_y + \frac{\partial A_x}{\partial z}V_z + \frac{\partial A_x}{\partial t}$$

Or,

$$\frac{dA_x}{dt} - \frac{\partial A_x}{\partial t} = \frac{\partial A_x}{\partial x}V_x + \frac{\partial A_x}{\partial y}V_y + \frac{\partial A_x}{\partial z}V_z$$

Putting this in equation 2 we get,

$$(\boldsymbol{V} \times \boldsymbol{\nabla} \times \boldsymbol{A})_x = \frac{\partial}{\partial x}(\boldsymbol{V}.\boldsymbol{A}) - \left[\frac{dA_x}{dt} - \frac{\partial A_x}{\partial t}\right]$$

Then equation 1 becomes

$$F_x = q\left[-\frac{\partial \phi}{\partial \mathrm{x}} - \frac{1}{C}\frac{\partial A_x}{\partial t} + \frac{1}{C}\left(\frac{\partial}{\partial x}(\boldsymbol{V}.\boldsymbol{A}) - \left[\frac{dA_x}{dt} - \frac{\partial A_x}{\partial t}\right]\right)\right]$$

$$= q\left[-\frac{\partial}{\partial \mathrm{x}}\left(\phi - \frac{1}{C}\boldsymbol{V}.\boldsymbol{A}\right) - \frac{1}{C}\frac{dA_x}{dt}\right]$$

Also

$$\frac{\partial}{\partial V_x}(\boldsymbol{V}.\boldsymbol{A}) = A_x\,, \qquad \text{since } \frac{\partial A_x}{\partial V_x} = 0$$

Thus

$$F_x = q\left[-\frac{\partial}{\partial \mathrm{x}}\left(\phi - \frac{1}{C}\boldsymbol{V}.\boldsymbol{A}\right) + \frac{d}{dt}\left\{\frac{\partial}{\partial V_x}\left(-\frac{\boldsymbol{V}.\boldsymbol{A}}{C}\right)\right\}\right]$$

Since ϕ is independent of velocity, we can write the term

$$\frac{\partial}{\partial V_x}\left(-\frac{\boldsymbol{V}.\boldsymbol{A}}{C}\right) \text{ as } \frac{\partial}{\partial V_x}\left(\phi - \frac{\boldsymbol{V}.\boldsymbol{A}}{C}\right)$$

Therefore,

$$F_x = q\left[-\frac{\partial}{\partial \mathrm{x}}\left(\phi - \frac{1}{C}\boldsymbol{V}.\boldsymbol{A}\right) + \frac{d}{dt}\left\{\frac{\partial}{\partial V_x}\left(\phi - \frac{\boldsymbol{V}.\boldsymbol{A}}{C}\right)\right\}\right]$$

Let us put

$$U = q\phi - \frac{q}{C}\boldsymbol{V}.\boldsymbol{A}$$

$$F_x = -\frac{\partial \mathrm{U}}{\partial \mathrm{x}} + \frac{d}{dt}\left(\frac{\partial U}{\partial V_x}\right) \tag{3}$$

It shows that the force acting on the particle is derivable from a potential which is dependent on velocity. Such a particle system is called <u>non-conservative</u> one and we apply the following equations of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \tag{4}$$

Writing equation (3) in generalized co-ordinates

$$F_x = Q_j = -\frac{\partial \mathrm{U}}{\partial \mathrm{q}_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

And putting in eqs. (4), we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial \mathrm{U}}{\partial \mathrm{q}_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

$$\frac{d}{dt}\left[\frac{\partial (T-U)}{\partial \dot{q}_j}\right] - \frac{\partial (T-U)}{\partial q_j} = 0$$

Which can be written as

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_j}\right] - \frac{\partial L}{\partial q_j} = 0$$

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Where Lagrangian $L = T - U = T - q\phi + \frac{q}{c}\boldsymbol{V}.\boldsymbol{A}$. This shows that form of Lagrange's equation even if the system is non-conservative.

Technique of Calculus of Variation

Solution of a dynamical problem means that, we want to locate the position of the system, e.g., a particle, at a particular instant of time. We are also interested in the path adopted by the system. The piece-wise information of this path i.e. where it is maximum or minimum is secured through differential calculus by putting $\dot{y}(x) = 0$ etc. But if we want the information about the whole path then we shall have to look for integral calculus and will be interested in the arguments like whether the path as a whole is largest or shortest (extremum or having a stationary value). This study requires the technique of calculus of variations. (fig. 1). Two paths which a particle may follow in going from position 1 to position 2 are shown. The straight-line path is shortest and can be represented as

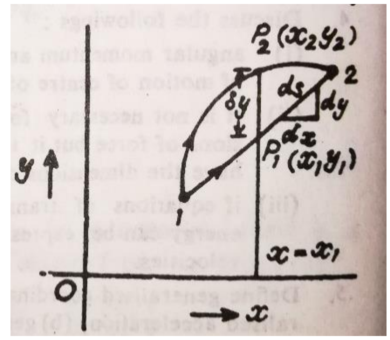

Fig. 1

$$y = mx + c$$

or
$$y = y(x)$$

in functional form indicating that y is a function of independent parameter x. For each value of x there will be fixed value of y. As we are interested in the length of the path, denoted by say, I, we can Write

$$I = \int ds = \int \sqrt{(dx^2 + dy^2)}$$

$$= \int \sqrt{\left(1 + \frac{dy^2}{dx^2}\right) dx^2}$$

$$\int \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} dx$$

$$= \int \sqrt{(1 + \dot{y}^2)} dx$$

$$= \int f(\dot{y}) dx$$

$$= \int f(y, \dot{y}, x) dx$$

Because $\dot{y}$ involves both y and x, we have expressed it in f. Thus, we note that path, I, is the integral of the function, f, which itself is the function of y. If we want that path, I, be extremum, any variation,δ, in it, i.e., δI should be zero. That is

$$\delta I = \delta \int f(y, \dot{y}, x) dx = 0$$

$$= \delta \int f[y(x), \dot{y}(x), x]dx = 0$$

This is the formulation of the problem of calculus of variations. Note that variation δ is defined as the variation in the quantity to which it is applied at the fixed value of independent parameter i.e.

$$\delta y = (y_2 - y_1)_x$$

If we account for both curves then the eq. y=y (x) is incapable to represent both. y=mx+ c will represent straight line path only. Therefore, to include other paths, we require another parameter, say α, designating path, to be introduced in y i.e.

$$y = y(x, \alpha)$$

So that, $$y = y(x, \alpha_1)$$

and $$y = y(x, \alpha_2)$$

may represent the two paths. However, the relationship can be expressed as

$$y(x, \alpha) = y(x, 0) + \alpha\eta(x)$$

Where, $\eta(x)$ is any arbitrary function of x which vanishes at end points. Note α=0 in y (x, 0) may be taken to represent extremum path. Thus α represents the paths. It means I, the path length, which is different for different paths, will also be a function of α, i.e., I=I(α) so that we may write the integral

$$\delta I(\alpha) = \delta \int f[y(x, \alpha), \dot{y}(x, \alpha), x]dx = 0$$

Thus, it may be noted that δ is taken at a fixed value of independent parameter, i.e., x is same for all paths considered i.e., x is not the function of α or we can write

$$\frac{\partial x}{\partial \alpha} = 0$$

Further, at endpoints, all paths meet and therefore there is no variation even in y coordinate at end points or

$$\left.\frac{\partial y}{\partial \alpha}\right|_{end\ points\ 1\ \&\ 2} = 0$$

The shortest path is taken as **extremum path** and the other paths with which we compare it (in our discussion of finding the extremum path) are called **comparison paths**.

In the variational approach, the actual physical behavior of the system is distinguished by the fact that it makes a certain integral functional stationary. Thus, all of the physics is somehow contained in the integrand of this functional!

Necessary Condition for the Integral $I = \int_{x_1}^{x_2} f(y, \dot{y}, x)dx$ to be an Extremum

We wish to find a function y (x) which will cause the integral

$$I = \int_{x_1}^{x_2} f(y, \dot{y}, x)dx \qquad (1)$$

to have a stationary value (extremum). The integral f is taken to be a function of the dependent variable y, independent variable x and $\dot{y} = \frac{dy}{dx}$. Thus, we write

$$\delta I = \delta \int_{x_1}^{x_2} f(y, \dot{y}, x)dx = 0$$

Where δ is called variation and represents the increase in the quantity to which it is applied in switching from stationary path to the comparison path at the fixed value of x.

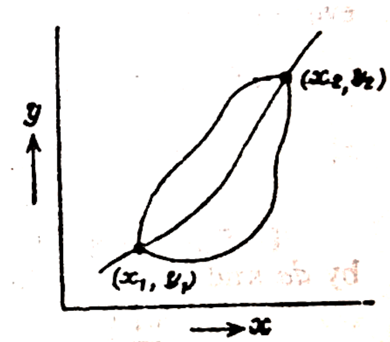

Fig. 1: Possible paths of motion

Since y(x) represents a path, our aim is to find a curve (path) between two points (x_1 and x_2) for which the integral is an extremum. We will, in solving the problem, take into account all the possible paths between the two points x_1 and x_2. We label all the possible curves y(x) with different value of a parameter α regarding that some value of α, say $\alpha = 0$, the curves would coincide with the path or paths for which the integral $\int f(y, \dot{y}, x)dx$ is extremum (may be taken as the shortest). Therefore, y should then be the function of both, x and the parameter α, i.e.

$$y(x, \alpha) = y(x, 0) + \alpha\eta(x)$$

Where $\eta(x)$ is any arbitrary function of x which vanishes at x = x_1 and x = x_2 since it is variation with fixed ends.

Thus, integral I, which will now be a function of α, can be written as

$$I(\alpha) = \int_{x_1}^{x_2} f[y(x, \alpha), \dot{y}(x, \alpha), x]dx$$

Since α refers to different paths, $\frac{\partial I}{\partial \alpha} = 0$ will correspond to a path for which the integral is an extremum. Thus

$$\frac{\partial I(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left\{\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha}\right\} dx$$

Since x is not a function of α, $\frac{\partial x}{\partial \alpha} = 0$ so that

$$\frac{\partial I(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left\{\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\right\} dx$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha} dx$$

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$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial y}{\partial x}\right)\right\} dx$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial \alpha \partial x} dx$$

$\left[\int uv dx = u \int v dx - \int \left(\frac{du}{dx}\int v dx\right) dx\right]$

Integrating the second term by parts,

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} dx + \left[\frac{\partial f}{\partial \dot{y}}\frac{\partial y}{\partial \alpha}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\frac{\partial y}{\partial \alpha} dx \tag{2}$$

Since the end points (x_1, y_1) and (x_2, y_2) are fixed and same for every curve so at end points x_1 and x_2 differentiation of different paths of motion vanishes, i.e. $\frac{\partial y}{\partial \alpha} = 0$, then $\left[\frac{\partial f}{\partial \dot{y}}\frac{\partial y}{\partial \alpha}\right]_{x_1}^{x_2} = 0$. Thus eq. (2) reduces to

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\right]\frac{\partial y}{\partial \alpha} dx$$

To find the stationary value, we multiply the above equation by $d\alpha$ and evaluate the derivative at $\alpha = 0$, so that

$$\left(\frac{\partial I}{\partial \alpha}\right)_0 d\alpha = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\right]\left(\frac{\partial y}{\partial \alpha}\right)_0 d\alpha dx \tag{3}$$

But $\left(\frac{\partial I}{\partial \alpha}\right)_0 d\alpha$ = increase in the integral I as we pass from the extremum path to the comparison path at the same value of x (i.e., δ-variation of I) = δI

Similarly, $\left(\frac{\partial y}{\partial \alpha}\right) d\alpha = \delta y$. Therefore equation (3) takes the form

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\right] \delta y dx$$

But δI = 0, then

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\right] \delta y dx = 0$$

Here, δy represents some arbitrary variation of y(x) with respect to the arbitrary parameter α about its extremum value (α=0). Since δy is arbitrary,

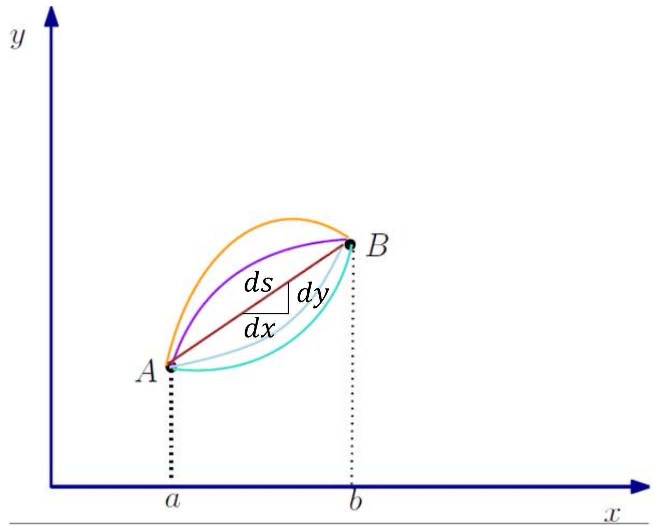
$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0 \tag{4}$$

Which is a relation that should be satisfied by a function f *[which is a function of y (x)]*, if the integral I is to be extremum.

FEW EXAMPLES

(A) Shortest distance between two points in a plane

A straight line is regarded as the shortest distance between two points in a plane. Thus, straight line is an extremum path of a particle in a plane and consequently, the equation of such a path should conveniently be obtained from the above technique of the calculus of variations.



An element of small are length *ds* in a plane can be represented as

$$ds = \sqrt{(dx^2 + dy^2)} = dx\sqrt{(1+\dot{y}^2)}$$

The total length of a curve between any two points 1 and 2 in a plane can be written as

$$I = \int_1^2 ds = \int_1^2 \sqrt{(1+\dot{y}^2)}dx = \int_1^2 f dx$$

For this curve to be shortest, $\delta I = 0$, i.e., the equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0 \qquad (1)$$

Where $f = \sqrt{(1+\dot{y}^2)}$ must be satisfied. Then $\frac{\partial f}{\partial y} = 0$. The equation (1) therefore reduces to

$$\frac{d}{dx}\left(\frac{\partial\sqrt{(1+\dot{y}^2)}}{\partial \dot{y}}\right) = 0$$

$$\frac{d}{dx}\left(\frac{1}{2}2\dot{y}(1+\dot{y}^2)^{-\frac{1}{2}}\right) = 0$$

$$\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{(1+\dot{y}^2)}}\right) = 0$$

$$\frac{\dot{y}}{\sqrt{(1+\dot{y}^2)}} = constant$$

$$\dot{y} = a$$

$$\frac{dy}{dx} = a$$

$$y = ax + b$$

for which $\dot{y} = \mathrm{a}$, a constant, giving y=ax+b. This is an equation representing a straight line; a and b can be calculated in terms of end points coordinates of the curve.

(B) Minimum surface of revolution

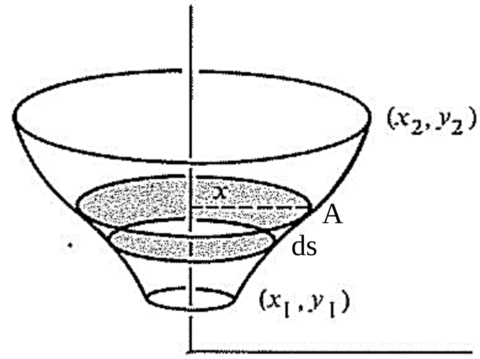


Fig. 1: Minimum surface of revolution

We form a surface of revolution by revolving a curve about a certain axis. In this example, curve passing through two end points (x_1, y_1) and (x_2, y_2) has been rotated about y-axis. Our aim is to find a curve which on revolving about a certain axis forms geometry of minimum surface area.

Let us consider a strip at point A formed due to the revolution of arc length ds about y axis. If the distance of this arc from y-axis be x, then surface area of the strip

$$= 2\pi x ds$$

$$= 2\pi x\sqrt{(1+\dot{y}^2)}dx$$

The total surface area is then

$$I = \int_1^2 2\pi x\sqrt{(1+\dot{y}^2)}dx$$

and will be minimum if $\delta I = 0$, for which the equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$$

should be satisfied. For this problem $f = x\sqrt{(1+\dot{y}^2)}$, so that

$$\frac{\partial f}{\partial y} = 0, \quad and \quad \frac{\partial f}{\partial \dot{y}} = \frac{x.2\dot{y}}{2\sqrt{(1+\dot{y}^2)}} = \frac{x\dot{y}}{\sqrt{(1+\dot{y}^2)}}$$

Putting in the said equation, we get

$$\frac{d}{dx}\left(\frac{x\dot{y}}{\sqrt{(1+\dot{y}^2)}}\right) = 0$$

$$\frac{x\dot{y}}{\sqrt{(1+\dot{y}^2)}} = a$$

a is a constant of integration. Squaring and arranging, we get

$$\frac{x^2\dot{y}^2}{1+\dot{y}^2} = a^2; \quad \Rightarrow x^2\dot{y}^2 = a^2 + a^2\dot{y}^2$$

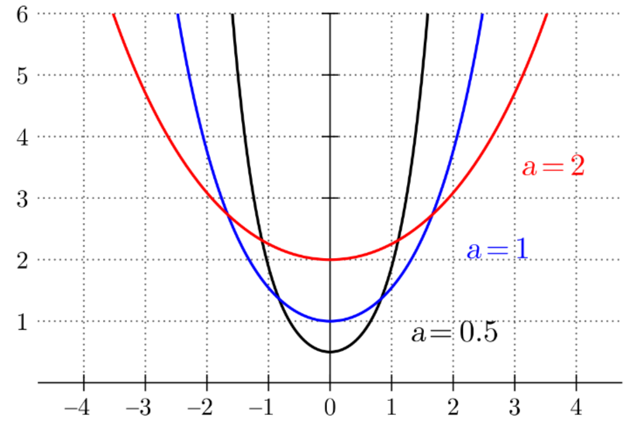
$$\dot{y}^2 = \frac{a^2}{x^2 - a^2} \quad \Rightarrow \dot{y} = \frac{a}{\sqrt{x^2-a^2}}$$

the integration of which yields

$$y = a\int \frac{dx}{\sqrt{x^2-a^2}} + b = a\, cosh^{-1}\frac{x}{a} + b$$

$$\Rightarrow x = a \cosh \frac{y-b}{a}$$

Which is the equation of a catenary *(the curve, that is an idealized hanging chain or cable assumes under its own weight when supported only at its ends. The catenary curve has a U-like shape, superficially similar in appearance to a parabolic arch, but it is not a parabola).*



Hamilton Principle (Variational Principle of least action):

The principle of least action – or, more accurately, the principle of stationary action – is a variational principle that, when applied to the action of a mechanical system, can be used to obtain the equations of motion for that system. It was historically called "least" because its solution requires finding the path of motion in space that has the least value. The principle can be used to derive Newtonian, Lagrangian and Hamiltonian equations of motion, and even general relativity.

Among all possible paths along which a dynamical system may move from one point to another point within specified time interval consistent with any constraints, the actual path followed is that which minimizes the time interval of the difference between the kinetic and potential energies.

In terms of calculus variations, Hamilton principle becomes,

$$\delta \int_{t_1}^{t_2} L dt = 0$$

Where, $L = T - U$ = Lagrangian function

$\delta =$ Variation symbol

The statement of the variation principle requires only that L be an extremum. The integral $\int_{t_1}^{t_2} L dt$ is denoted by I. i.e.

$$\mathrm{I} = \int_{t_1}^{t_2} L dt$$

Where I is referred to as an action of action integral.

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Lagrange's Equation of Motion from Hamilton's Principle for Conservative System:

According to Hamilton's variational principle, motion of a conservative system from time t_1 to time t_2 is such that the variation of the line integral $I = \int_{t_1}^{t_2} L[q_j(t),\ \dot{q}_j(t), t]dt$ is zero, i.e.,

$$\delta I = \delta \int_{t_1}^{t_2} L[q_j(t),\ \dot{q}_j(t), t]dt = 0 \tag{1}$$

Now we shall show that, Lagrange's equation of motion follows directly from Hamilton's principle. If we account for all possible paths of motion of the system in configuration space and label each with a value of a parameter α, then, since paths are being represented by $q_j(t,\alpha)$, I also becomes a function of α so that we write,

$$I(\alpha) = \int_{t_1}^{t_2} L[q_j(t,\alpha),\ \dot{q}_j(t,\alpha), t]dt$$

So that,

$$\frac{\partial I(\alpha)}{\partial \alpha} = \int_{t_1}^{t_2} \sum_j \left[\frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial \alpha} + \frac{\partial L}{\partial t}\frac{\partial t}{\partial \alpha}\right] dt \tag{2}$$

Since in δ variation, there is no time variation along any path and also at end points and hence $\frac{\partial t}{\partial \alpha}$ is zero along all paths. Therefore, on multiplying by $d\alpha$, above equation becomes

$$\frac{\partial I(\alpha)}{\partial \alpha} d\alpha = \int_{t_1}^{t_2} \sum_j \left[\frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial \alpha}\right] d\alpha dt$$

$$= \int_{t_1}^{t_2} \sum_j \frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial \alpha} d\alpha dt + \int_{t_1}^{t_2} \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{\partial^2 q_j}{\partial t \partial \alpha} d\alpha dt$$

Integrating second term by parts,

$$= \int_{t_1}^{t_2} \sum_j \frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial \alpha} d\alpha dt + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{\partial q_j}{\partial \alpha} d\alpha \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\frac{\partial q_j}{\partial \alpha} d\alpha dt$$

The middle term is zero since δ variation involves fixed end points. Then $\sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{\partial q_j}{\partial \alpha} d\alpha \bigg|_{t_1}^{t_2} = 0$ at end points t_2 and t_1

$$\therefore \qquad \frac{\partial I(\alpha)}{\partial \alpha} d\alpha = \int_{t_1}^{t_2} \sum_j \frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial \alpha} d\alpha dt - \int_{t_1}^{t_2} \sum_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\frac{\partial q_j}{\partial \alpha} d\alpha dt$$

$$\Rightarrow \qquad \delta I(\alpha) = \int_{t_1}^{t_2} \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\right] \delta q_j\, dt$$

Since q_j are independent of each other, the variations δq_j will be independent. Hence $\delta I(\alpha) = 0$ if and only if the coefficient of δq_j separately vanishes, i.e.,

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

Which are Lagrange's equations of motions for a conservative system. It is thus obvious that these equations follow directly from Hamilton's principle.

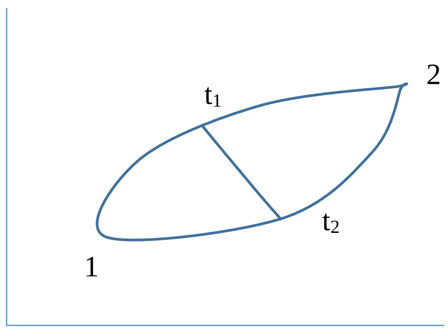
Lagrange's Equations Using Variational Principle for Non-Conservative Systems Involving Forces not Derivable from Potential Functions:

To include the non-conservative forces, let us extend the principle to assume a form

$$\delta I = \delta \int_1^2 (T + W) dt = 0$$

$$\Rightarrow \int_1^2 \delta T dt + \int_1^2 \delta W dt = 0 \qquad (1)$$

With fixed end points, where $\delta W = \delta \sum \boldsymbol{F_i}.\boldsymbol{r_i} = \sum \boldsymbol{F_i}.\delta \boldsymbol{r_i}$ represents the work done by the force on the system during the virtual displacement from actual to the rest conceived or rather varied paths. The variation δ doesnot include time variation. Thus, the time of motion for the system along every path is same.



Possible paths are referred to as $q_j(t,\ \alpha)$. The transformation equations can be written as

$$\boldsymbol{r}_i = \boldsymbol{r}_i[q_j(t,\alpha), t]$$

From which we find that,

$$\delta \boldsymbol{r}_i = \sum_j \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j$$

Further, the components of generalized force are expressed as

$$\delta W = \sum_i \boldsymbol{F_i}.\delta \boldsymbol{r_i}$$

$$= \sum_{i,j} \boldsymbol{F_i}.\frac{\partial \boldsymbol{r_i}}{\partial q_j} \delta q_j$$

$$= \sum_j Q_j \delta q_j$$

Because, $Q_j = \sum_i \boldsymbol{F_i} . \frac{\partial \boldsymbol{r_i}}{\partial q_j}$. Thus equation 1 takes the form

$$\int_1^2 \delta T(q_j, \dot{q}_j) dt + \int_1^2 \sum_j Q_j \delta q_j \, dt = 0$$

$$\int_1^2 \sum_j \left(\frac{\partial T}{\partial q_j} \delta q_j + \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j \right) dt + \int_1^2 \sum_j Q_j \delta q_j \, dt = 0 \qquad (2)$$

$$\int_1^2 \sum_j \left(\frac{\partial T}{\partial q_j} \delta q_j \right) dt + \frac{\partial T}{\partial \dot{q}_j} \delta q_j \Big|_1^2 - \int_1^2 \sum_j \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \delta q_j \right) dt + \int_1^2 \sum_j Q_j \delta q_j \, dt = 0$$

$\left[\int uv dx = u \int v dx - \int \left(\frac{du}{dx} \int v dx \right) dx \right]$

The middle term is zero as it is a variation with fixed end point. So

$$\int_1^2 \sum_j \left[\frac{\partial T}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) + Q_j \right] \delta q_j dt = 0$$

Again, since the constraints are assumed to be holonomic, δq_j are independent of each other, and the above integral can vanish if and only if the coefficients of δq_j separately vanishes, i.e.

$$\frac{\partial T}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) + Q_j = 0$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j$$

Which are Lagrange's equations of motion for non-conservative system.

Again, we know

$$Q_j = \frac{\partial V}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial V}{\partial \dot{q}_j} \right)$$

Then, from second term of eqn 2 of LHS can be written as

$$\int_1^2 \sum_j Q_j \delta q_j \, dt = - \int_1^2 \sum_j \delta q_j \left[\frac{\partial V}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial V}{\partial \dot{q}_j} \right) \right] dt \qquad (3)$$

$\left[\int uv dx = u \int v dx - \int \left(\frac{du}{dx} \int v dx \right) dx \right]$

Integrating by parts of second term of RHS of equation 3 we get

$$\int_1^2 \sum_j \frac{d}{dt} \left(\frac{\partial V}{\partial \dot{q}_j} \right) \delta q_j \, dt = \sum_j \frac{\partial V}{\partial \dot{q}_j} \delta q_j \Big|_1^2 - \int_1^2 \sum_j \left(\frac{\partial V}{\partial \dot{q}_j} \right) \delta \dot{q}_j \, dt$$

$$= - \int_1^2 \sum_j \left(\frac{\partial V}{\partial \dot{q}_j} \right) \delta \dot{q}_j \, dt \qquad (4)$$

From eqn. 3 and 4 we get,

$$\int_1^2 \sum_j Q_j \delta q_j \, dt = -\int_1^2 \sum_j \left[\frac{\partial V}{\partial q_j} \delta q_j - \left(\frac{\partial V}{\partial \dot{q}_j} \right) \delta \dot{q}_j \right] dt$$

$$= -\delta \int_1^2 V dt \qquad [As\ V = V(q, \dot{q})] \qquad (5)$$

Hence, eqn.2 becomes

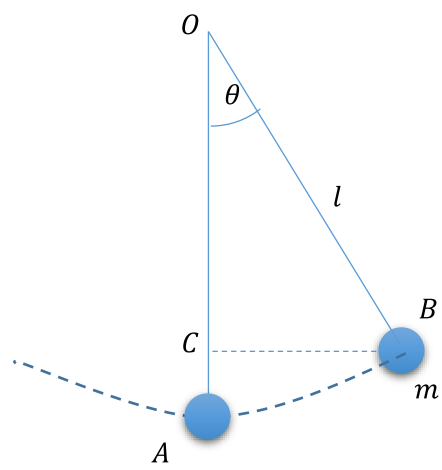
$$\delta I = \int_1^2 \delta T dt - \int_1^2 \delta V dt$$

$$= \delta \int_1^2 (T - V) dt$$

Applications of Lagrange's Equation of Motion:

1) Simple Pendulum

The angle θ between rest position and deflected position is chosen as generalized coordinate. If the string is of length l, then kinetic energy is



$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (l\dot{\theta})^2 = \frac{1}{2} m l^2 \dot{\theta}^2$$

Where m is the mass of the bob.

In coming from position B to A, the mass has fallen freely through a vertical distance CA. Thus potential energy is

$$V = mg(OA - OC)$$

$$= mg(l - l cos\theta)$$

$$= mgl(1 - cos\theta)$$

Where the reference level or zero level of potential energy has been taken at a distance *l* below the point of suspension. Thus Lagrangian is

$$L = T - V$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - cos\theta)$$

So that $\qquad \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}$

And $\qquad \frac{\partial L}{\partial \theta} = -mgl \sin\theta$

Putting in Lagrange's equation $\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

we get,

$$\frac{d}{dt}\left(ml^2\dot{\theta}\right) + mgl\sin\theta = 0$$

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Which is an equation of simple harmonic motion with time period

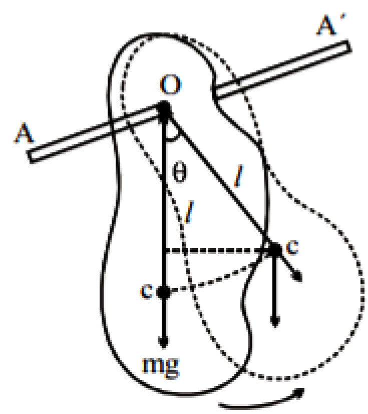
$$T = 2\pi\sqrt{\frac{l}{g}}$$

2) Compound Pendulum

A rigid body capable of oscillating in a vertical plane above a fixed horizontal axis is called a compound pendulum. Let vertical plane of oscillation be xy, O be the point through which axis of rotation passes, C is the center of mass. Let mass of the pendulum be m, moment of inertia about axis of rotation I and distance OC=*l*.

If θ be the angle through which the body is defrelted, then kinetic energy is

$$T = \frac{1}{2}I\dot{\theta}^2$$



The potential energy relative to a horizontal plane through O is

$$V = -mgl\ cos\theta$$

The Lagrangian is

$$L = T - V = \frac{1}{2}I\dot{\theta}^2 + mgl\ cos\theta$$

So that,

$$\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

As θ is chosen as generalized coordinate, Lagrange's equation is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$$\frac{d}{dt}\left(I\dot{\theta}\right) + mgl\sin\theta = 0$$

$$I\ddot{\theta} + mgl\sin\theta = 0$$

$$\ddot{\theta} + \frac{mgl}{I}\sin\theta = 0$$

If amplitude of oscillation is small then

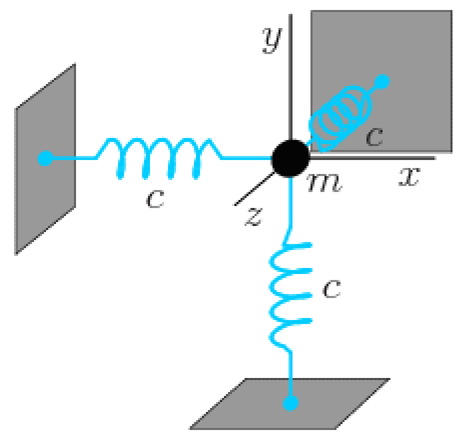
$$\ddot{\theta} + \frac{mgl}{I}\theta = 0$$

Which is an equation of simple harmonic motion of time period

$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

2) Isotropic Oscillator (three dimensional):

An isotropic harmonic oscillator can be considered as a vibrating particle that is acted upon by a force directed always towards or away from the position of equilibrium and the magnitude of which varies linearly with the distance from the position of equilibrium. Such a force can be represented as $F=-kr$ where k is called the force constant. The potential energy is



$$V = -\int F dr$$

$$= -\int -kr dr$$

$$= \frac{1}{2}kr^2 + c$$

If we choose the horizontal plane passing through the position of equilibrium as the reference level, then V=0 at r=0, making the constant, c, zero. Thus

$$V = \frac{1}{2}kr^2$$

$$= \frac{1}{2}k(x^2 + y^2 + z^2)$$

The kinetic energy of the oscillator is

$$T = \frac{1}{2}m\dot{r}^2$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

The Lagrangian L for the oscillator can be written as

$$L = T - V$$

Which in three dimensions, on using Cartesian coordinates, takes the form,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}k(x^2 + y^2 + z^2) \quad (1)$$

Giving,

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial L}{\partial \dot{y}} = m\dot{y} \qquad \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$\frac{\partial L}{\partial x} = -kx \qquad \frac{\partial L}{\partial y} = -ky \qquad \frac{\partial L}{\partial z} = -kz \qquad (2)$$

The Lagrange's equation in Cartesian coordinates are

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \end{aligned}\right\} \qquad (3)$$

Or,

$$\left.\begin{aligned} m\ddot{x} + kx = 0 \\ m\ddot{y} + ky = 0 \\ m\ddot{z} + kz = 0 \end{aligned}\right\} \qquad (4)$$

This gives the desired equations of motion for three-dimensional isotropic oscillator.

Conservation of Energy:

Let us consider:

i) a conservative system, so that, the potential energy is a function of coordinates only and not that of velocities,

ii) constraints do not change with time, i.e., they are independent of time and consequently, equation of transformation to generalized coordinates do not involve time explicitly and hence

iii) L can be written as $L(q_{j,}\dot{q}_j)$

Thus, its total time derivative will be

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\frac{dq_j}{dt} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} \qquad (1)$$

Putting $\frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$ from Lagrange's equation, we get

$$\frac{dL}{dt} = \sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\frac{dq_j}{dt} + \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt}\right]$$

$$= \sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt}\right]$$

$$= \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) \tag{2}$$

Because for a conservative system (velocity independent potential)

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} = p_j \quad as \quad \frac{\partial V}{\partial \dot{q}_j} = 0$$

Thus, from above equation

$$\frac{dL}{dt} - \sum_j \frac{d}{dt}(\dot{q}_j p_j) = 0$$

$$\frac{d}{dt}\left(L - \sum_j (\dot{q}_j p_j)\right) = 0$$

$$\frac{d}{dt}\left(\sum_j (\dot{q}_j p_j) - L\right) = 0$$

$$\boldsymbol{\sum_j (\dot{q}_j p_j) - L = constant = H} = Hamiltonian\ of\ the\ system \tag{3}$$

Now, Euler's theorem on homogeneous function states that if ***f*** is a homogeneous function of order n, of a set of variable q_j, then,

$$\left[e.g.\ f = T = \frac{1}{2}mv^2 \quad here\ n = 2\right]$$

$$\sum_j q_j \frac{\partial f}{\partial q_j} = nf$$

In our case, here n is 2, so that

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T$$

$$\sum_j \dot{q}_j p_j = 2T$$

Thus from eqn. 3,

$$H = 2T - L$$

$$H = 2T - (T - V)$$

$$H = T + V$$

Which shows that H equals total energy and it is constant i.e. conserved.