Assignment for Honours Part-III, Examination/2020

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Simple interest rate

For the interest rate r the value $V(T)$ at time T of holding P units of currency starting at time $t = 0$ is $V(T) = (1 + rT)P$, where T is expressed in years.

Compound interest rate

For the interest rate r the value $V(T)$ at time T of holding P units of currency starting at time $t = 0$ is $V(T) = (1 + \frac{r}{m})^{mT}P$, where m is the number interest payments made per annum.

Continuous compounding

For a constant interest rate r the time value of money under continuous compounding is given by $V(T) = e^{rT}P$.

Return

Let us denote the asset price at time t by $S(t)$. The meaningful quantity for the change of an asset price is its relative change $\frac{\Delta S}{S}$, which is called the return, where $\Delta S = S(t+\delta t) - S(t)$. In other words,

$$\text{Return} = \frac{\text{change in value over a period of time}}{\text{initial investment}}$$

.

In the limit $\delta t \to 0$, its becomes $\frac{dS}{S}$.

Introduction

Simple Model for Stock Price

$$\frac{dS}{S} = \mu dt + \sigma dW$$

Deterministic part: This can be modeled by $\frac{dS}{S} = \mu dt$. Here, μ is a measure of the growth rate of the asset. We may think μ is a constant during the life of an option.

Random part: The second part is a random change in response to external effects,such as unexpected news. It is modeled by a Brownian motion σdW, the σ is the order of fluctuations or the variance of the return and is called the volatility. The quantity σdW is sampled from a normal distribution.

In other words σdW describes the stochastic change in the share price, where dW stands for $dW = W(t + \delta t) - W(t)$ as $\delta t \to 0$, $W(t)$ is a Wiener process, σ is the volatility.

The Brownian Motion (Wiener Precess)

A time dependent function $W(t), t \in \mathbf{R}$ is said to be a Brownian motion if
(a) For all t, $W(t)$ is a random variable, i.e.

$$W(0) = 0.$$

(b) $W(t)$ has continuous path, i.e. $W(t)$ is continuous in t.
(c) $W(t)$ has independent increments. For any $u > 0$, $v > 0$ the increments $W(t+u) - W(t)$ and $W(t+v) - W(t)$ are independent.
(d) For all $\sigma > 0$, the increments $W(t+\sigma) - W(t)$ is normaly distributed with mean zero and variance σ, i.e.

$$W(t+\sigma) - W(t) \sim N(0, \sigma).$$

Normal Distribution

The probability density function for a random variable $W(t)$ has a normal distribution with mean μ and variance σ^2, then the probability density function is

$$pW(t) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right), -\infty < t < \infty.$$

The probability density function for a random variable $W(z)$ has a standard normal distribution with mean 0 and variance 1, i.e., $W(z) \sim N(0, 1)$ then the probability density function is

$$pW(z) = \frac{1}{2\pi} exp\left(-\frac{z^2}{2}\right).$$

Properties

Show that

$$(i) \qquad E[W] = 0;$$
$$(ii) \qquad E[W^2] = t;$$
$$(iii) \qquad E[\Delta W] = 0;$$
$$(iv) \qquad E[(\Delta W)^2] = \Delta t;$$
$$(v) \qquad \Delta W = X(\Delta t)^{(1/2)}, \text{where} X \sim N(0,1)$$

Properties

(i) $E[W(t)] = \int_a^b tW(t)dt$, for $a \leq W(t) \leq b$.

$$= \int_{-\infty}^{\infty} \frac{t}{\sqrt{2\pi\sigma}} e^{-\frac{t^2}{2\sigma}} dt, \text{ put } z = \frac{t^2}{2\sigma}$$
$$= \frac{\sqrt{\sigma}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z} dz = 0.$$

(ii) $E[W(t)^2] = \int_{-\infty}^{\infty} \frac{t^2}{\sqrt{2\pi\sigma}} e^{-\frac{t^2}{2\sigma}} dt$ put $z = \frac{t}{\sqrt{\sigma}}$.

$$=> \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} dz = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} zd(e^{-\frac{z^2}{2}})$$
$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = \frac{\sigma}{\sqrt{2\pi}} \sqrt{2\pi} = \sigma.$$

Question

1. Show that the return $\frac{\Delta S}{S}$ is normally distributed with mean $\mu \Delta t$ and variance $\sigma^2 \Delta t$.
2. Show that $\Delta S \sim N(\mu S \Delta t, \sigma^2 S^2 \Delta t)$.
3. Show that $\ln S(t)$ is a normal distribution with mean $\ln S_0 + (\mu - \frac{\sigma^2}{2})t$ and variance $\sigma^2 t$.

Answer

1. We have $\frac{\Delta S}{S} = \mu\Delta t + \sigma\Delta W = \mu\Delta t + \sigma(W(t+\Delta t) - W(t))$.
But $\mu\Delta t + \sigma\Delta W \sim N(0, \Delta t) => \Delta W = X\sqrt{\Delta t}$, where $X \sim N(0,1)$.
Therefore, $\frac{\Delta S}{S} = \mu\Delta t + \sigma X\sqrt{\Delta t}$, $X \sim N(0,1)$
$=> E(\frac{\Delta S}{S}) = \mu\Delta t + \sigma\sqrt{\Delta t}E(X) = \mu\Delta t + \sigma\sqrt{\Delta t}\cdot 0 = \mu\Delta t$.
Hence, mean=$\mu\Delta t$ Again, $V(\frac{\Delta S}{S}) = \sigma^2\Delta t V(X) = \sigma^2\Delta t \cdot 1$.
Hence, variance=$\sigma^2\Delta t$
2. In similar fashion, we can prove question (2).
3. Solution of question (3)?

How to calculate small changes in a function that is dependent on the values determined by stochastic differential equation ?

Let $f(S)$ be the desired smooth function of S; since f is sufficiently smooth we know that small changes in the asset's price, dS, result in small changes to the function f. Recall that we approximated df with a Taylor series expansion, resulting in

$$df = \frac{df}{dS}dS + \frac{1}{2}\frac{d^{v}f}{dS^2}dS^2 + ...; \text{ but } dS^2 = (\mu S dt + \sigma S dW)^2$$

$$=> df = \frac{df}{dS}(\mu S dt + \sigma S dW) + \frac{1}{2}\frac{d^2 f}{dS^2}(\mu S dt + \sigma S dW)^2 + ...$$

Assumption: As $dt \to 0, dW = \sigma(\sqrt{dt}) => dW/vdt = 1$ and $dWdt = o(dt) => dWdt = 0.$

Implies that $dS^2 \to \sigma^2 S^2 dt$ as $dt \to 0.$

How to calculate small changes in a function that is dependent on the values determined by stochastic differential equation ?

$$
\begin{aligned}
df &= \frac{df}{dS}(\mu S dt + \sigma S dW) + \frac{1}{2}\frac{d^2 f}{dS^2}(\sigma S dW)^2. \\
&= \frac{df}{dS}(\mu S dt + \sigma S dW) + \frac{1}{2}\frac{d^2 f}{dS^2}(\sigma^2 S^2 dt). \\
&= (\mu S \frac{df}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2})dt + \sigma S \frac{df}{dS} dW.
\end{aligned}
$$

Itô's Lemma

Statement: For any function $f(S,t)$ of two variables W and t where S satisfies Stochastic Differential Equation (SDE) $dS = \mu dt + \sigma dW$ for some constants μ and σ, $dW(t)$ is a Brownian motion (Wiener Process), then the general form of Itô's Lemma is

$$df = (\frac{df}{dt} + \mu S\frac{df}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2})dt + \sigma S\frac{df}{dS}dW.$$

Now consider f to be a function of both S and t. So long as we are aware of partial derivatives, we can once again expand our function (now $f(S+dS, t+dt)$) using a Taylor series approximation about (S,t) to get:

$$df = \frac{df}{dS}dS + \frac{df}{dt}dt + \frac{1}{2}(\frac{d^2 f}{dS^2}dS^2 + \frac{d^2 f}{dt^2}dt^2 + 2\frac{d^2 f}{dSdt}dSdt) + ...$$

Example

Show that $\ln S(t)$ is a normal distribution with mean $\ln S_0 + (\mu - \frac{\sigma^2}{2})t$ and variance $\sigma^2 t$.

solution: We apply Itos lemma with $x = f(S) = \ln S$

$$\begin{aligned} dx &= (0 + \frac{1}{S}.\mu S - \frac{1}{2}.\frac{1}{S^2}.\sigma^2 S^2)dt + \frac{1}{S}.\sigma S dW \\ &= (\mu + \frac{\sigma^2}{2})dt + \sigma dW => dx(t) = (\mu - \frac{\sigma^2}{2})dt + \sigma dW \end{aligned}$$

$$=> x(t) - x(0) = (\mu - \frac{\sigma^2}{2})t + \sigma W(t) \sim N((\mu + \frac{\sigma^2}{2})t, \sigma^2 t)$$

$$=> \ln S(t) = \ln S(0) + (\mu - \frac{\sigma^2}{2})t + \sigma W(t) \sim N(\ln S(0) + (\mu - \frac{\sigma^2}{2})t, \sigma^2 t)$$

Geometric Brownian motion

By above example

$$\ln S(t) = \ln S(0) + (\mu - \frac{\sigma^2}{2})t + \sigma W(t)$$
$$=> \ln(\frac{S(t)}{S(0)}) = (\mu + \frac{\sigma^2}{2})t + \sigma W(t)$$
$$=> S(t) = S(0) exp\{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)\}$$

The transition probability density function for $S(t)$ is

$$P(S(t) = S|S(0) = S_0) = \frac{1}{\sqrt{2\pi\sigma^2 t}S} e^{-(\ln(S/S_0 - (\mu - \frac{\sigma^2}{2})t)^2/2\sigma^2 t)}$$

This is called the log-normal distribution.

Quetion

Show that the mean and variance of the geometric Brownian motion are:
(a) $E[S(t)] = \int_{-\infty}^{\infty} sP_{S(t)}(s)ds = S_0 e^{\mu t}$,
(b) $Var[S(t)] = S_0^2 e^{2\mu t}[e^{\sigma^2 t} - 1]$.
Proof:(a) $E[S(t)] = \int_0^{\infty} sP_{S(t)}(s)ds$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-(\ln(S/S_0 - (\mu - \frac{\sigma^2}{2})t)^2/2\sigma^2 t)} dS$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^x e^{(x - x_0 - (\mu - \frac{\sigma^2}{2})t)^2/2\sigma^2 t)} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{x + x_0 + \mu t} e^{(x + \frac{\sigma^2}{2} t)^2/2\sigma^2 t} dx$$

$$= S_0 e^{\mu t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{x - (x + \frac{\sigma^2}{2} t)^2/2\sigma^2 t} dx$$

$$= S_0 e^{\mu t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{(x - \frac{\sigma^2}{2} t)^2/2\sigma^2 t} dx$$

$$= S_0 e^{\mu t}$$

Proof:(b) $E[S(t)^2] = \int_0^{\infty} \frac{S}{\sqrt{2\pi\sigma^2 t}} e^{-(\ln(S/S_0-(\mu-\frac{\sigma^2}{2})t)^2/2\sigma^2 t)} dS$
$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{2x} e^{(x-x_0-(\mu-\frac{\sigma^2}{2})t)^2/2\sigma^2 t)} dx$
$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{2(x+x_0+\mu t)} e^{(x+\frac{\sigma^2}{2}t)^2/2\sigma^2 t} dx$
$= S_0^2 e^{2\mu t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{2x-(x+\frac{\sigma^2}{2}t)^2/2\sigma^2 t} dx$
$= S_0^2 e^{\mu t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-(x-\frac{3\sigma^2}{2}t)^2/2\sigma^2 t+\sigma^2 t} dx$
$= S_0^2 e^{2\mu t+\sigma^2 t}$. Therefore, $Var(S(t)) = E[S(t)^2] - E[S(t)]^2 = S_0^2 e^{2\mu t}[e^{\sigma^2 t} - 1]$

Definition

European call option gives the holder the right (not obligation) to buy the underlying asset at a prescribed time T (expiry date/maturity) for a specified (exercise/strike) price X.
European put option gives its holder the right (not obligation) to sell underlying asset at the expiry time T for the exercise price X.

Europian Options

Call Option

A call option gives the holder the right (but not the obligation) to buy the underlying asset by a certain date for a certain price.

The price in the contract is known as the exercise or strike price (denoted by X)

The date in the contract is known as the expiration or exercise or maturity date (denoted by T)

The price of the underlying stock at the expiration date is denoted by $S(T)$.

$$Payoff = \begin{cases} S(T) - X & if\ S(T) > X \\ 0 & otherwise. \end{cases}$$

At time $0 < t < T$: Call Premium/Profit=Payoff-Call value=$max(S(T) - X, 0) - C$.

At time T: Gain of the buyer for a call is $max(S(T) - X, 0) - Ce^{rT}$.

Europian Options

Europian options can only be exercise at the expiration date.

Put Option

A put option gives the holder the right (but not the obligation) to sell the underlying asset by a certain date (T) for a certain price (X). In expiration if the stock price $(S(T))$.

$$Payoff = \begin{cases} X - S(T) & if\ X > S(T) \\ 0 & otherwise. \end{cases}$$

At time $0 < t < T$: Put Premium/Profit=Payoff-Put value=$max(X - S(T), 0) - P$.

At time T: Gain of the seller for a put is $max(X - S(T), 0) - Pe^{rT}$.

Portfolios and Short Selling

Portfolios

Portfolio is a combination of assets, options and bonds.

We denote by V the value of a portfolio. Example: $V = 2S + 4C - 5P$. It means that the portfolio consists of long position(+) in two shares, long position(+) in four call options and a short position (-) in five put options.

Short Selling

Short selling is the practice of selling assets that have been borrowed from a broker with the intention of buying the same assets back at a later date to return to the broker.

This technique is used by investors who try to profit from the falling price of a stock.

Trading Strategies

Straddle

Straddle is the purchase of a call and a put on the same underlying security with the same maturity time T and strike price X. The value of portfolio is $V = C + P$

Straddle is effective when an investor is confident that a stock price will change dramatically, but is uncertain of the direction of price move.

Short Straddle, $V = -C - P$, profits when the underlying security changes little in price before the expiration $t = T$.

Bull Spread

Bull spread is a strategy that is designed to profit from a moderate rise in the price of the underlying security.

Let us set up a portfolio consisting of a long position in call with strike price X_1 and short position in call with X_2 such that $X_1 < X_2$. The value of this portfolio is $V_t = C_t(X_1) - C_t(X_2)$. At maturity $t = T$

$$V_T = \begin{cases} 0 & S \leq X_1 \\ S - X_1 & X_1 \leq S < X_2 \\ X_2 - X_1 & S \geq X_2 \end{cases}$$

Bond Pricing

Bond

A Bond is a contract that yields a known amount F, called the face value, on a known time T, called the maturity date. The authorised issuer (for example, government) owes the holder a debt and is obliged to repay the face value at maturity and may also pay interest (the coupon).

Zero-coupon bond

A Zero-coupon bond does not pay any coupons and involves only a single payment at T.

No Arbitrage Principle

One of the key principles of financial mathematics is the No Arbitrage Principle.

There are never opportunities to make risk-free profit.

Arbitrage opportunity arises when a zero initial investment $V_T = 0$ is identified that guarantees non-negative payoff in the future such that $V_T > 0$ with non-zero probability.

Arbitrage opportunities may exist in a real market. But, they cannot last for a long time.

All risk-free portfolios must have the same rate of return.

Let V be the value of a risk-free portfolio, and dV is its increment during a small period of time dt. Then

$$\frac{dV}{dt} = rdt,$$

where r is the risk-free interest rate.

Let V_t be the value of the portfolio at time t. If $V_T = 0$, then $V_t = 0$ for $t < T$.

Upper Bound of a Call $S(t)$

Consider two portfolio having:
Portfolio A: one Stock
Portfolio B: one Call
At time $0 < t < T$, Value of
Portfolio A: $S(t)$
Portfolio B: C
At time T, Value of
Portfolio A: $S(T)$
Portfolio B: $max(S(T) - X, 0)$
Now, at time T, value of B: $max(S(T) - X, 0)$
$= -min(X - S(T), 0) - S(T) + S(T)$
$= -min(X, S(T)) + S(T)$
Since $-min(X, S(T)) + S(T) < S(T) => \text{Value of} A > \text{Value of} B$
Therefore, at time t, Value of$A$ $>$ Value of$B$ by no arbitrage principle.
$=> S(t) > C$.
Hence, $S(t)$ is upper bound of a Call.

Lower Bound of a Call $C \geq S(t) - Xe^{-r(T-t)}$

Consider two portfolio having:
Portfolio A: one Call+Cash
Portfolio B: one Stock
At time $0 < t < T$, Value of
Portfolio A: $C + Xe^{-r(T-t)}$
Portfolio B: $S(t)$
At time T, Value of
Portfolio A: $max(S(T) - X, 0) + X$
Portfolio B: $S(T)$
Now, at time T, value of A: $max(S(T) - X, 0) + X$
$= max(S(T), X)$
$= -min(X, S(T)) - S(T) + S(T)$
Since $-min(X, S(T)) + S(T) < S(T) =>$ Value of$A \geq$ Value of$B$
Therefore, at time t, Value of$A \geq$ Value of$B$ by no arbitrage principle.
$=> C + Xe^{-r(T-t)} \geq S(t)$.
Hence, $C \geq S(t) - Xe^{-r(T-t)}$ is lower bound of a Call.

Upper Bound of a Put $P \leq Xe^{-r(T-t)}$

Consider two portfolio having:

Portfolio A: one put

Portfolio B: some cash

At time $0 < t < T$, Value of

Portfolio A: P

Portfolio B: $Xe^{-r(T-t)}$

At time T, Value of

Portfolio A: $max(X - S(T), 0)$

Portfolio B: X

Now, at T, $V(B) = X = X + max(X - S(T), 0) - max(X - S(T), 0)$
$= max(X - S(T), 0) + min(S(T), X) \geq max(X - S(T), 0)$ Therefore, at T, $V(B) \geq V(A) => Xe^{-r(T-t)} \geq P$

Lower Bound of a Put $P \geq Xe^{-r(T-t)} - S(t)$

Consider two portfolio having:
Portfolio A: one put+one stock
Portfolio B: some cash
At time $0 < t < T$, Value of
Portfolio A: $P + S(t)$
Portfolio B: $Xe^{-r(T-t)}$
At time T, Value of
Portfolio A: $max(X - S(T), 0) + S(T)$
Portfolio B: X
Now, at T, $V(A) = max(X - S(T), 0) + S(T)$
$= max(X, S(T)) \geq X$ Therefore, at T, $V(A) \geq V(B)$
$=> P + S(t) \geq Xe^{-r(T-t)}$

Call-Put Parity $C - P = S(0) - Xe^{-rT}$

Consider two portfolio having:

Portfolio A: one call+ Cash $Xe^{-r(T-t)}$

Portfolio B: one put+one Stock

At time $t = 0$, Value of

Portfolio A: $C + Xe^{-rT}$

Portfolio B: $P + S(0)$

At time T, Value of

Portfolio A: $max(S(T) - X, 0) + X = max(S(T), X)$

Portfolio B: $max(X - S(T), 0) + S(T) = max(X, S(T))$

Since, at T, $V(A) = V(B)$

Therefore, $C - P = S(0) - Xe^{-rT}$

Dependance of C^E and P^E on X

Suppose that $X' < X''$, let $X' + a = X''$, where $a > 0$
At time T,
$C^E(X') = max(S(T) - X', 0)$
$C^E(X'') = max(S(T) - X'', 0)$
$= max(S(T) - X', 0) - max(S(T) - a, 0)$
Since, at T,$max(S(T) - X', 0) > max(S(T) - X', 0) - max(S(T) - a, 0)$
Therefore, $C^E(X') > C^E(X'')$

Quetion: Show that if $X' < X''$ then
$C^E(X') - C^E(X'') < e^{-rT}(X'' - X')$
$P^E(X') - P^E(X'') < e^{-rT}(X'' - X')$
solution: From put-call parity

Dependance of C^E and P^E on $S(0)$

Suppose that stock price increases $S' < S''$, let $S' + a = S''$, where $a > 0$

At time T,

$C^E(S') = max(S'(T) - X, 0)$

$C^E(S'') = max(S''(T) - X, 0)$

$= max(S'(T) - X, 0) + max(a - X, 0)$

Since, at T, $max(S'(T) - X, 0) + max(a - X, 0) > max(S'(T) - X, 0)$

Therefore, $C^E(S'') > C^E(S')$

Question: Show that if $S' < S''$ then

$C^E(S'') - C^E(S') < (S'' - S')$

$P^E(S') - P^E(S'') < (S'' - S')$

American Option

American options can be exercised at any time up to the expiration date.

Dependance of C^A and P^A on X.

Consider two portfolio having:

Portfolio A: one call with strike price X'

Portfolio B: one call with strike price X''

At time $t = 0$, Value of

Portfolio A: $C^A(X')$

Portfolio B: $C^A(X'')$

At time $0 < t < T$, value of

Portfolio A: $C^A(X')e^{-r(T-t)} = max(S(T) - X', 0)e^{-r(T-t)}$

Portfolio B: $C^A(X'')e^{-r(T-t)} = max(S(T) - X'', 0)e^{-r(T-t)}$

Suppose that $X' < X''$, let $X' + a = X''$, where $a > 0$

The value of B: $max(S(T) - X'', 0)e^{-r(T-t)}$

$= max(S(T) - X', 0)e^{-r(T-t)} - max(S(T) - a, 0)e^{-r(T-t)} => V(A) > V(B)$

$=> C^A(X') > C^A(X'')$.

Dependance of C^A and P^A on $S(t)$

Consider two portfolio having:
Portfolio A: one call with stock price S'
Portfolio B: one call with stock price S''
At time $t = 0$, Value of
Portfolio A: $C^A(S')$
Portfolio B: $C^A(S'')$
At time $0 < t < T$, value of
Portfolio A: $C^A(S')e^{-r(T-t)} = max(S'(T) - X, 0)e^{-r(T-t)}$
Portfolio B: $C^A(S'')e^{-r(T-t)} = max(S''(T) - X, 0)e^{-r(T-t)}$
Suppose that $S' < S''$, let $S' + a = S''$, where $a > 0$
The value of B: $max(S''(T) - X, 0)e^{-r(T-t)}$
$= max(S'(T) - X, 0)e^{-r(T-t)} - max(S'(T) - a, 0)e^{-r(T-t)}$
$=> V(B) > V(A)$
$=> C^A(S'') > C^A(S')$.

Dependance of C^A and P^A on expiry T

Consider two portfolio having:

Portfolio A: one call with expiry time T'

Portfolio B: one call with expiry time T''

At time $0 < t < expiry$, Value of

Portfolio A: $max(S(T') - X, 0)e^{-r(T'-t)}$

Portfolio B: $max(S(T'') - X, 0)e^{-r(T''-t)}$

Suppose that $T' < T''$, let $T' + t = T''$, where $t > 0$

Now, $V(B) = max(S(T' + t) - X, 0)e^{-r(T'+t-t)}$

$= max(S(T') - X, 0)e^{-rT'} + max(S(t) - X, 0)e^{-rT'}$

Again, $V(A) = max(S(T') - X, 0)e^{-r(T'-t)}$

$= e^{rT'} max(S(T') - X, 0)e^{-rT'}$

At time $t = 0$,

$V(A) = max(S(T') - X, 0)e^{-rT'}$

$V(B) = max(S(T') - X, 0)e^{-rT'} + max(S(0) - X, 0)e^{-rT'}$

$=> V(B) > V(A)$

$=> C^A(T'') > C^A(T')$.

Questions

1. Show that if $X' < X''$ then
$C^A(X') - C^A(X'') < (X'' - X')$
$P^A(X'') - P^A(X') < (X'' - X')$

2. Show that if $S' < S''$ then
$C^A(S'') - C^A(S') < (S'' - S')$
$P^A(S') - P^A(S'') < (S'' - S')$

3. Show that if $T' < T''$ then
$C^A(T'') - C^A(T') < (T'' - T')$
$P^A(T') - P^A(T'') < (T'' - T')$

Binomial distribution

Notation: $X \sim \text{Binomial}(n, p)$.

Description: number of successes in n independent trials, each with probability p of success. Probability function:

$$f_X(x) = P(X = x) = \binom{n}{x} P^x (1-p)^{n-x} \text{ for } x = 0, 1, ..., n.$$

Mean: $E(X) = np$.

Variance: $Var(X) = np(1-p) = npq$, where $q = 1 - p$.

Sum: if $X \sim \text{Binomial}(n, p), Y \sim \text{Binomial}(m, p)$,
then $X + Y \sim Bin(n + m, p)$.

Table of Binomial Model

$S(0)$	$S(0)u$	$S(0)u^2$	$S(0)u^3$
	$S(0)d$	$S(0)ud$	$S(0)u^2d$
		$S(0)d^2$	$S(0)ud^2$
			$S(0)d^3$

$Period : S(0)$	$S(1)$	$S(2)$	$S(3)$
$S(0)$	S^u	S^{uu}	S^{uuu}
	S^d	S^{ud}	S^{uud}
		S^{dd}	S^{udd}
			S^{ddd}

$S^u = S(0)u$, $S^{uu} = S^u u = S(0)uu$, $S^{uuu} = S^{uu}u = S^u uu = S(0)uuu$

One Step Binomial Model

Derivative for a risk-neutral valuation:

We set up a portfolio consisting of a long position in Δ shares S and short position of the cash bond B. Then $D = \Delta S - B$ In the next period, the portfolio has one of two possible values:

1. $\Delta S^u - Be^{rdt}$ or,
2. $\Delta S^d - Be^{rdt}$

We want to duplicate the values of derivatives by our portfolio as a function as

1. $\Delta S^u - Be^{rdt} = f(S^u)$ or,
2. $\Delta S^d - Be^{rdt} = f(S^d)$

The solution of Δ and B are

$$\Delta = \frac{f(S^u) - f(S^d)}{S^u - S^d}, \text{ and } B = -e^{-rdt}\left(f(S^u) - \frac{f(S^u) - f(S^d)}{S^u - S^d}S^u\right)$$

One Step Binomial Model

Since, $S^u = S(0)u$, $S^d = S(0)d$

$$\Delta = \frac{f(S^u) - f(S^d)}{S(0)(u-d)}, \text{ and } B = -e^{-rdt}\left(f(S^u) - \frac{f(S^u) - f(S^d)}{(u-d)}u\right)$$

$$=> \Delta = \frac{f(S^u) - f(S^d)}{S(0)(u-d)}, \text{ and } B = \left(\frac{df(S^u) - uf(S^d)}{e^{-rdt}(u-d)}\right)$$

The initial value of derivative should be the same as that of the portfolio $V_0 = \Delta S(0) - B$, which is

$$\begin{aligned} D(0) &= \frac{f(S^u) - f(S^d)}{(u-d)} - \left(\frac{df(S^u) - uf(S^d)}{e^{rdt}(u-d)}\right) \\ &= \frac{(f(S^u) - f(S^d))e^{rdt} - df(S^u) + uf(S^d)}{e^{rdt}(u-d)} \end{aligned}$$

One Step Binomial Model

$$D(0) = \frac{(e^{rdt} - d)f(S^u) + (u - e^{rdt})f(S^d)}{e^{rdt}(u-d)}$$

$$= \frac{1}{e^{rdt}}[(\frac{e^{rdt} - d}{u-d})f(S^u) + (1 - \frac{e^{rdt} - d}{u-d})f(S^d)]$$

$$f(S(1)) = \frac{1}{e^{rdt}}\{p_* f(S^u) + (1-p_*)f(S^d)\} \textit{ where } p_* = \frac{e^{rdt} - d}{u-d}$$

Two cases occurs: $p_* = \frac{e^{rdt}-d}{u-d}$

(i) Assume $u > d$, if $p_* \leq 0$ then $e^{rdt} \leq d$ then $e^{rdt} \leq d < u$. Then for a greened risk free profit buy stock and sell cash bound.

(ii) if $p_* \geq 1$ then $d < u \leq e^{rdt}$. Then for a greened risk free profit sell stock and buy cash bound.

Conclusion:

The probability of an up movement in the stock price occurs when $0 \leq p_* \leq 1$.

Two-Steps Binomial Model

In the two-steps Binomial model option expires after two time steps as S_{uu}, S_{ud} and S_{dd}.
Let $f(S^u)$ and $f(S^d)$ be the options

$$f(S^u) = \frac{p_* S^{uu} + (1 - p_*) S^{ud}}{e^{r\delta t}}, \quad f(S^d) = \frac{p_* S^{ud} + (1 - p_*) S^{dd}}{e^{r\delta t}}$$

Now,

$$\begin{aligned} f(S(2)) &= \frac{p_* f(S^u) + (1 - p_*) f(S^d)}{e^{r\delta t}} \\ &= \frac{p_*(p_* S^{uu} + (1 - p_*) S^{ud}) + (1 - p_*)(p_* S^{ud} + (1 - p_*) S^{dd})}{e^{2r\delta t}} \\ &= \frac{p_*^2 S^{uu} + 2p_*(1 - p_*) S^{ud} + (1 - p_*)^2 S^{dd}}{e^{2r\delta t}} \end{aligned}$$

Multi-Steps Binomial Model

For two-steps Binomial model

$$D(0) = f(S(2)) = \frac{p_*^2 S^{uu} + 2p_*(1-p_*)S^{ud} + (1-p_*)^2 S^{dd}}{e^{2r\delta t}}$$

$$\begin{aligned} D(0) &= f(S(1)) = e^{-rdt} f(S(1)) \\ &= f(S(2)) = e^{-2rdt} f(S(2)) \\ &= f(S(3)) = e^{-3rdt} f(S(3)) \end{aligned}$$

Multi-Steps Binomial Model

If $S(1) => $ single step, $S(2) => $ two steps, $S(3) => $ three steps then

$$f(S(1)) = p_* f(S^u) + (1 - p_*) f(S^d)$$
$$f(S(2)) = p_*^2 f(S^{uu}) + 2p_*(1 - p_*) f(S^{ud}) + (1 - p_*)^2 f(S^{dd})$$
$$f(S(3)) = p_*^3 f(S^{uuu}) + 3p_*^2(1 - p_*) f(S^{uud}) + 3p_*(1 - p_*)^2 f(S^{udd})$$
$$+(1 - p_*)^3 f(S^{ddd}).$$

$$D(0) = \Delta S(0) + B, \quad + \mathit{or} \ - \ \mathit{sign\ gives\ same\ result.}$$
$$= e^{-rdt}\{p_* f(S^u) + (1 - p_*) f(S^d)\}$$
$$= e^{-2rdt}\{p_*^2 f(S^{uu}) + 2p_*(1 - p_*) f(S^{ud}) + (1 - p_*)^2 f(S^{dd})\}$$
$$= e^{-3rdt}\{p_*^3 f(S^{uuu}) + 3p_*^2(1 - p_*) f(S^{uud}) + 3p_*(1 - p_*)^2 f(S^{udd})$$
$$+(1 - p_*)^3 f(S^{ddd})\}$$

$$
\begin{aligned}
V(0) &= \Delta S(0) - B = \frac{1}{e^{r\delta t}}\{pC_u + (1-p)C_d\} \\
&= \frac{1}{e^{2r\delta t}}\{p^2C_{uu} + 2p(1-p)C_{ud} + (1-p)^2C_{dd}\} \\
&= \frac{1}{e^{3r\delta t}}\{p^3C_{uuu} + 3p^2(1-p)C_{uud} + 3p(1-p)^2C_{udd} + (1-p)^3C_{dda} \\
&= \frac{1}{e^{3r\delta t}}\sum_{j=0}^{3}\binom{3}{j}p^j(1-p)^{3-j}(u^jd^{3-j}S - X)^+. \\
&= \frac{1}{e^{nr\delta t}}\sum_{j=0}^{n}\binom{n}{j}p^j(1-p)^{n-j}(u^jd^{n-j}S - X)^+.
\end{aligned}
$$

If the price of $f(0)$ is not equal to $D(0)$, the arbitrage opportunity exists. If $f(0) > D(0)$ the trader can short the derivative (get amount of money equal to $f(0)$), and buy the portfolio (pay $D(0)$, and has some profit. At the expiry, the trader can sell the portfolio and buy the derivative (to return it for the short selling). Therefore, the profit $f(0) - D(0)$ is risk-free.

Risk-Neutral valuation

We consider a portfolio consisting of a long position in Δ shares and short position in one call, then $V = \Delta S - C$

- By no-arbitrage arguments we derive the current call option price is $C_0 = \Delta S_0 - (\Delta S_0 u - C_u)e^{-rT}$,
- We can interpret a Risk-Neutral valuation by taking $C_0 = e^{-rT}(pC_u + (1-p)C_d)$
- Our subjective probability of up movement p does not appear in the final formula.
- This is because $V_T = K$ same value on up or down movement.
- The value P appears in the formula and can be thought of as a probability.

Risk-Neutral valuation

We consider a portfolio consisting of a long position in Δ shares and short position in one call, then $V = \Delta S - C$

- It is the probability implied by the market.
- Fair price of a call option C_0 is equal to the expected value of its future payoff discounted at the risk-free interest rate.
- For a put option P_0 (or in fact any financial contract) we have the same result $P_0 = e^{-rT}(pP_u + (1-p)P_d)$.
- We interpret the variable $0 \leq p \leq 1$ as the probability of an up movement in the stock price. This formula is known as a risk-neutral valuation.
- The probability of up p or down movement $1-p$ in the stock price plays no role.