Wave Filters

Two-port network

A two-port network (a kind of four-terminal network or **quadripole**) is an electrical network (circuit) or device with two pairs of terminals to connect to external circuits. Two terminals constitute a port if the currents applied to them satisfy the essential requirement known as the port condition: the electric current entering one terminal must equal the current emerging from the other terminal on the same port. The ports constitute interfaces where the network connects to other networks, the points where signals are applied or outputs are taken. In a two-port network, often port 1 is considered the input port, and port 2 is considered the output port.

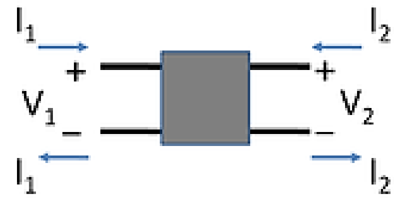


Figure 1: Example two-port network with symbol definitions. Notice the port condition is satisfied: the same current flows into each port as leaves that port.

The two-port network model is used in mathematical circuit analysis techniques to isolate portions of larger circuits. A two-port network is regarded as a "black box" with its properties specified by a matrix of numbers. This allows the response of the network to signals applied to the ports to be calculated easily, without solving for all the internal voltages and currents in the network. It also allows similar circuits or devices to be compared easily. For example, transistors are often regarded as two-ports, characterized by their h-parameters (see below) which are listed by the manufacturer. Any linear circuit with four terminals can be regarded as a two-port network provided that it does not contain an independent source and satisfies the port conditions.

Examples of circuits analyzed as two-ports are filters, matching networks, transmission lines, transformers, and small-signal models for transistors (such as the hybrid-pi model). The analysis of passive two-port networks is an outgrowth of reciprocity theorems first derived by Lorentz.

In two-port mathematical models, the network is described by a 2 by 2 square matrix of complex numbers. The common models that are used are referred to as z-parameters, y-parameters, h-parameters, g-parameters, and ABCD-parameters, each described individually below. These are all limited to linear networks since an underlying assumption of their derivation is that any given circuit condition is a linear superposition of various short-circuit and open-circuit conditions. They are usually expressed in matrix notation, and they establish relations between the variables

V_1 , voltage across port 1

I_1, current into port 1

V_2 , voltage across port 2

I_2, current into port 2

which are shown in figure 1. The difference between the various models lies in which of these variables are regarded as the independent variables. These current and voltage variables are most useful at low-to-moderate frequencies. At high frequencies (e.g., microwave frequencies), the use of power and energy variables is more appropriate, and the two-port current-voltage approach is replaced by an approach based upon scattering parameters.

The port conditions

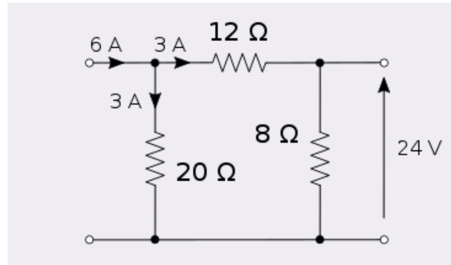
The port condition is that a pair of poles of a circuit is considered a port if and only if the current flowing into one pole from outside the circuit is equal to the current flowing out of the other pole into the external circuit. Equivalently, the algebraic sum of the currents flowing into the two poles from the external circuit must be zero

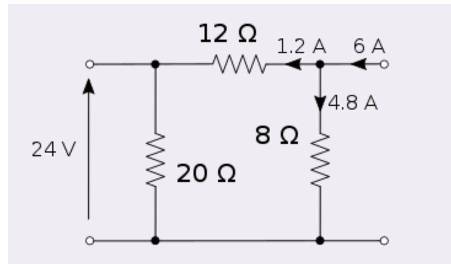
General properties

There are certain properties of two-ports that frequently occur in practical networks and can be used to greatly simplify the analysis. These include:

Reciprocal networks

A network is said to be reciprocal if the voltage appearing at port 2 due to a current applied at port 1 is the same as the voltage appearing at port 1 when the same current is applied to port 2. Exchanging voltage and current results in an equivalent definition of reciprocity. A network that consists entirely of linear passive components (that is, resistors, capacitors and inductors) is usually reciprocal, a notable exception being passive circulators and isolators that contain magnetized materials. In general, it will not be reciprocal if it contains active components such as generators or transistors.





Reciprocity in electrical networks is a property of a circuit that relates voltages and currents at two points. The reciprocity theorem states that the current at one point in a circuit due to a voltage at a second point is the same as the current at the second point due to the same voltage at the first. The reciprocity theorem is valid for almost all passive networks.

Symmetrical networks

A network is symmetrical if its input impedance is equal to its output impedance. Most often, but not necessarily, symmetrical networks are also physically symmetrical. Sometimes also antimetrical networks are of interest. These are networks where the input and output impedances are the duals of each other.

Lossless network

A lossless network is one, that contains no resistors or other dissipative elements.

Attenuation

Attenuation in an electrical system is the loss or reduction in the amplitude or strength of a signal as it passes along its length or some electric network. As the signal travels through the copper wire conductor some of the signals will be absorbed.

Attenuation is a result of resistance in the conductor and associated dielectric losses which are exaggerated by longer run lengths and higher frequency signals. By improving the dielectric properties of the insulation and increasing the conductor size it will reduce the attenuation.

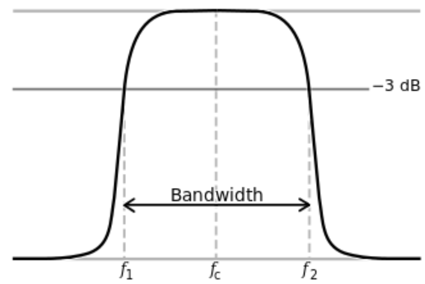
Attenuation, an amplitude loss, usually measured in dB, experienced by a signal after passing through a filter. Filter attenuation is the ratio, at a given frequency, of the signal amplitude at the output of the filter over the signal amplitude at the input of the filter, defined as

$$Attenuation = 10\ log_{10}\frac{P_0}{P_i}\ dB = 20\ log_{10}\frac{V_0}{V_i}\ dB$$

Cut of Frequency

The cutoff frequency, corner frequency, or break frequency is a boundary in a system's frequency response at which energy flowing through the system begins to be reduced (attenuated or reflected) rather than passing through.

Typically, in electronic systems such as filters and communication channels, cutoff frequency applies to an edge in lowpass, highpass, bandpass, or band-stop characteristic – a frequency characterizing a boundary between a passband and a stopband. It is sometimes taken to be the point in the filter response where a transition band and passband meet, for example, as defined by a half-power point (a frequency for which the output of the circuit is −3 dB of the nominal passband value). Alternatively, a stopband corner frequency may be specified as a point where a transition band and a stopband meet: a frequency for which the attenuation is larger than the required stopband attenuation, which for example may be 30 dB or 100 dB.



Magnitude transfer function of a bandpass filter with lower 3 dB cutoff frequency f_1 and upper 3 dB cutoff frequency f_2

In the case of a waveguide or an antenna, the cutoff frequencies correspond to the lower and upper cutoff wavelengths.

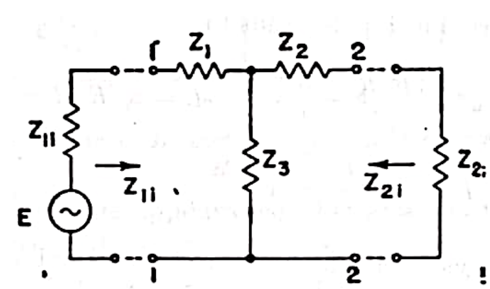
Characteristic Impedance

When two series arms of a T network are equal or the shunt arms of a π network are equal, the network is said to be symmetrical. For a symmetrical network the image impedances are equal to each other and the image impedance is then called the characteristic impedance or the iterative impedance.

It is defined as the particular value of the load **impedance** which can produce an input **impedance** with the value as same as the value of the load **impedance**. In the **two-port** system when it is connected at the one end then it produces equal **impedance** when looking at each other.

Image Impedance

Consider a T section of impedances interposed between a generator having internal impedance Z_{1i} and a load of impedance Z_{2i}, as in. It is desired that the impedance at the 1,1 terminals, into which the generator supplies power, be equal to the generator impedance, and that the impedance looking into the 2,2 terminals be equal to the load Z_{2i}. Under these conditions the impedance at 1,1 looking in one direction is the image of the impedance looking in the other direction, and Z_{1i} is called an image impedance of the network. Likewise, at 2,2 the impedance looking in one direction is the same as that looking in the other, so that Z_{2i} is also an image impedance at the 2,2 terminals. The network is then said to be matched on an image basis.



The values of the image impedances of the T section may be computed. The impedance Z_{1in} at the 1,1 terminals is required to be Z_{1i} and is

$$Z_{1in} = Z_{1i} = Z_1 + \frac{Z_3(Z_2 + Z_{2i})}{Z_3 + Z_2 + Z_{2i}}$$

Likewise, the impedance looking into the 2,2 terminals is required to be Z_{2i}, and is

$$Z_{2i} = Z_2 + \frac{Z_3(Z_1 + Z_{1i})}{Z_3 + Z_1 + Z_{1i}}$$

In general, the image impedances of ports 1 and 2 will not be equal unless the network is symmetrical (or anti-symmetrical) with respect to the ports. For a symmetrical network, $Z_{1i} = Z_{2i} = Z_0 = Characteristic\ Impedance$.

Propagation Constant/Transmission Function/Propagation Function/Transmission Parameter

The propagation constant, symbol γ, for a given system is defined by the ratio of the complex amplitude at the source of the wave to the complex amplitude at some distance *x*, such that,

$$\frac{A_0}{A_x} = e^{\gamma x}$$

Since the propagation constant is a complex quantity, we can write:

$$\gamma = \alpha + j\beta$$

Where,

- α, the real part, is called the attenuation constant $\left(\frac{A_0}{A_x} = e^{\alpha x}\right)$
- β, the imaginary part, is called the phase constant $\left(\beta = \frac{2\pi}{\lambda}\right)$

The propagation constant of a sinusoidal electromagnetic wave is a measure of the change undergone by the amplitude and phase of the wave as it propagates in a given direction. The quantity being measured can be the voltage, the current in a circuit, or a field vector such as electric field strength or flux density. The propagation constant itself measures the change per unit length, but it is otherwise dimensionless. In the context of two-port networks and their cascades, propagation constant measures the change undergone by the source quantity as it propagates from one port to the next.

The propagation constant's value is expressed logarithmically, almost universally to the base e, rather than the more usual base 10 that is used in telecommunications in other situations. The quantity measured, such as voltage, is expressed as a sinusoidal phasor. The phase of the sinusoid varies with distance which results in the propagation constant being a complex number, the imaginary part being caused by the phase change.

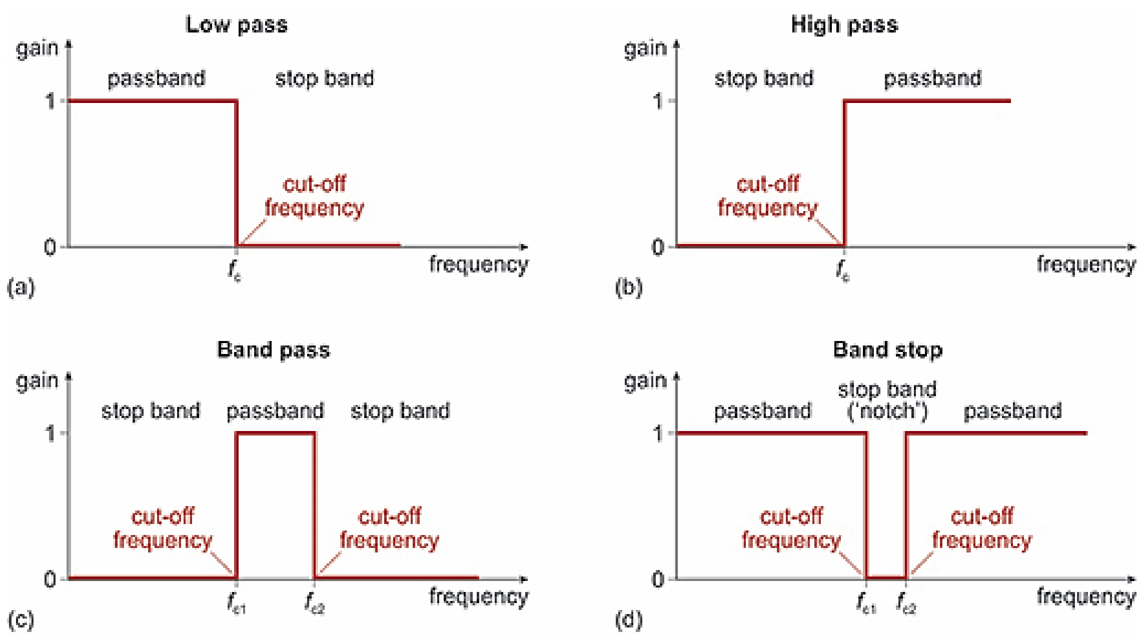
Filter Circuit:

Filters are electrical networks used to separate alternating from direct current components or to separate a group of A.C. components included within a particular frequency range from those lying outside this range. So a filter can be defined as a network that in its ideal form has at least one range of frequency in which the attenuation is zero (**pass hand**) and at least one range of frequency in which the attenuation is infinite (**attenuation band**). The frequencies which separate a pass band and attenuation hand are called **cut-off frequencies**. To achieve the desired effect, the filter is designed to provide a low attenuation for frequency components within a particular pass band range and a high attenuation at frequencies within other stop band ranges. The networks provide a uniform response over a wide range of frequencies than that obtained with resonant circuits. Filters are commonly classified in accordance with their selectivity characteristics as below :

(a) **A low pass filter.** It transmits all frequencies below a limiting frequency f_c. known as cut-off frequency. and stops all these above this frequency.

(b) **A high pass filter**. It passes frequencies above the cut-off frequency and stops all those below this frequency.
(c) **A band pass filter**. It passes frequencies in a particular hand between two cut-off frequencies and stops those above and below this band limit.

(d) **A band elimination filter**. It stops frequencies within a specified band and passes those above and below the units of this hand.



Hyperbolic Function

It is assumed that the student is familiar with some of the properties of hyperbolic functions, at least for real angles. Hyperbolic angles also have geometric meaning, being related to a hyperbola in the same way that trigonometric functions are related to a circle. This is illustrated in Fig.1, wherein the hyperbola is the locus for the radius r, and

$$sinh\,u = \frac{a}{r},\quad cosh\,u = \frac{b}{r},\quad tanh\,u = \frac{a}{b}$$

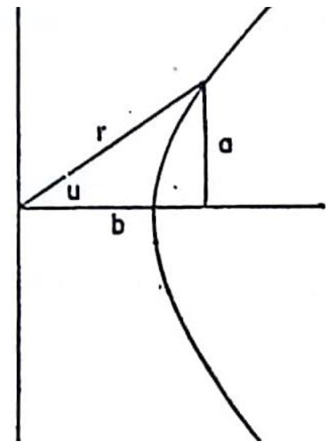


Fig. 1

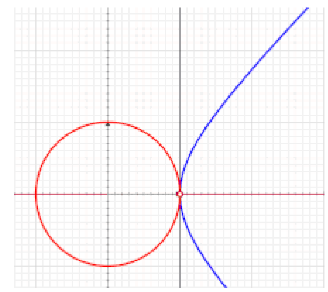
As they will be used here, hyperbolic functions simplify the writing of certain exponential relations, and knowledge of their limits is particularly useful. A few properties are here summarized and extended to the case of complex angles: the so-called "imaginary" angles which are called "hyperbolic" in contrast with ordinary "real" angles which are called "circular," combinations of these two kinds of angles are called "general" or "complex."

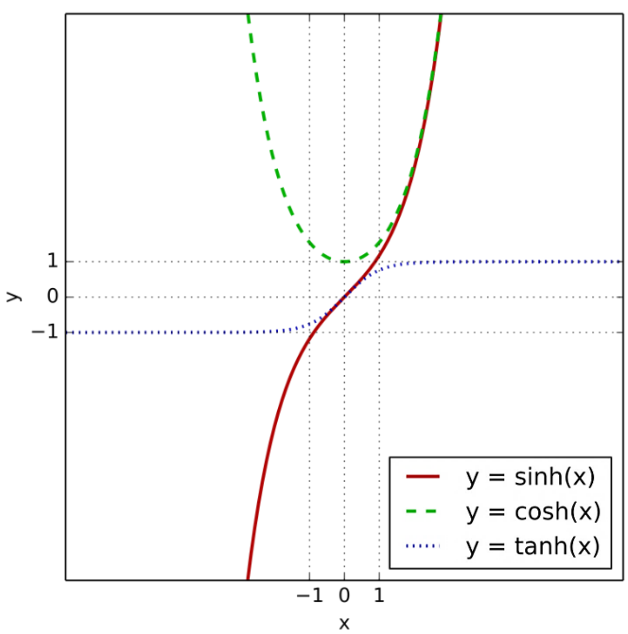
$$sinh\,u = \frac{e^u - e^{-u}}{2}$$

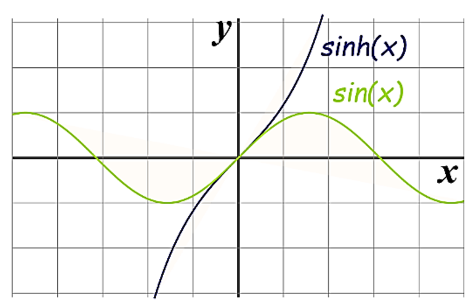
$$cosh\,u = \frac{e^u + e^{-u}}{2}$$

$$tanh\,u = \frac{sinh\,u}{cosh\,u} = \frac{e^u - e^{-u}}{e^u + e^{-u}}$$

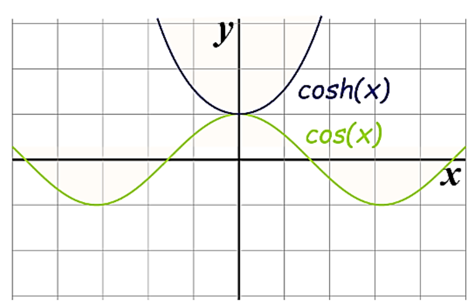
$$cosh^2 u - sinh^2 u = 1$$







sinh vs sin



cosh vs cos

values of the functions at the limits $u = 0$, and $u = \infty$ are

	$u = 0$	$u = \infty$
$sinh\, u$	0	∞
$cosh\, u$	1	∞
$tanh\, u$	0	1

For u large, $sinh\, u = cosh\, u$. If a is imaginary or $u = jw$, then

$$sinh\, jw = \frac{e^{jw} - e^{-jw}}{2} = j\, sin\, w$$

$$cosh\, jw = \frac{e^{jw} + e^{-jw}}{2} = cos\, w$$

Expressions for complex angles, where $u = a + jb$, can be obtained by expansions:

$$sinh(a + jb) = sinh\, a\, cosh\, jb + cosh\, a\, sinh\, jb$$

$$= sinh\, a\, cos\, b + j\, cosh\, a\, sin\, b$$

$$cosh(a + jb) = cosh\, a\, cosh\, jb + sinh\, a\, sinh\, jb$$

$$= cosh\, a\, cos\, b + j\, cosh\, a\, sin\, b$$

A few useful half-angle identities, which can be proved from the above are:

$$sinh\frac{u}{2} = \sqrt{\frac{1}{2}(cosh\, u - 1)}$$

$$cosh\frac{u}{2} = \sqrt{\frac{1}{2}(cosh\, u + 1)}$$

$$sinh\, u = 2\, sinh\frac{u}{2} cosh\frac{u}{2}$$

A considerable number of hyperbolic functions will prove useful in the sections to follow.

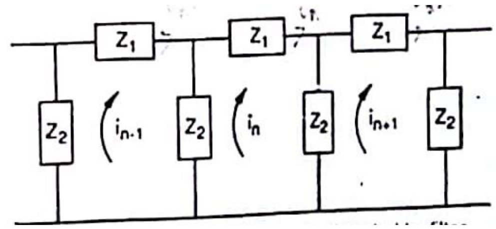
$$sin\, x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$cos\, x = \frac{e^{jx} + e^{-jx}}{2}$$

$$tan\, x = \frac{e^{jx} - e^{-jx}}{j(e^{jx} + e^{-jx})}$$

Elementary Filter Theory

The network consisting of sections each with an impedance Z_1 in the series arm and Z_2 in the shunt arm constitute the electric wave filter. Z_1 and Z_2 can either be inductances or capacitances or a combination of both.



Because of the wave nature of voltages and currents, their evaluation in different elements by the application of Kirchhoff's law would be very laborious. For a simple treatment. we proceed in a different way making use of the recurrent nature of the elements.

Let us consider a uniform filter, consisting of a chain of similar sections as shown in fig., which are repeated indefinitely forming an infinite chain. If a generator is applied at some point earlier in the chain, currents will flow in various sections. Let the currents in successive sections be i_{n-1}, i_n, i_{n+1}. Application of Kirchhoff's law to the central section gives the following expression:

$$Z_1 i_n + Z_2(i_n - i_{n+1}) - Z_2(i_{n-1} - i_n) = 0$$

$$-Z_2 i_{n-1} + (Z_1 + 2Z_2) i_n - Z_2 i_{n+1} = 0 \qquad (1)$$

We must write $i_n = a\, i_{n-1}$. where a is a real or complex number which is the attenuation constant. Then in an infinite chain where we cannot distinguish between sections, we must have $i_{n+1} = a\, i_n = a^2\, i_{n-1}$.

Then equation (1) gives

$$-Z_2 i_{n-1} + (Z_1 + 2Z_2) a\, i_{n-1} - Z_2 a^2\, i_{n-1} = 0$$

$$-Z_2 + (Z_1 + 2Z_2) a - Z_2 a^2 = 0$$

$$1 + \left(\frac{Z_1}{-Z_2} - 2\right) a + a^2 = 0 \qquad \text{[Dividing by } -Z_2\text{]}$$

$$a^2 - 2\left(\frac{Z_1}{2Z_2} + 1\right) a + 1 = 0 \qquad (2)$$

The equation determines the attenuation constant a. <u>We shall confine to the case where $\frac{Z_1}{Z_2}$ is real</u>. Which corresponds to Z_1 and Z_2, being both pure resistances or pure reactances. Then a can be either real or complex, but not a purely imaginary quantity. The roots will be real when $\frac{Z_1}{4Z_2}$ lies outside the range $0\ to\ -1$. We consider separately the three cases where it is greater than 0, between 0 and -1, and less than -1.

Discriminant Notes

$$\boldsymbol{D = b^2 - 4ac}$$

$D > 0$	2 real roots
$D = 0$	1 real root
$D < 0$	2 complex roots

The discriminant of this equation is

$$D = 4\left(\frac{Z_1}{2Z_2} + 1\right)^2 - 4$$

$$= 4\frac{{Z_1}^2}{4{Z_2}^2} + 4.2.\frac{Z_1}{2Z_2} + 4 - 4$$

$$= \frac{{Z_1}^2}{{Z_2}^2} + 4\frac{Z_1}{Z_2}$$

Case-1: ***a*** **is real and positive.**

Since the network does not contain power generating elements, the currents must decrease as we move away from the generator attached to one end of the filter. The significance of the positive sign of a is that the wave is attenuated without change of phase. If we write $a = e^{-\alpha}$, where i α is the attenuation constant per section, equation (2) becomes

$$e^{-2\alpha} - 2\left(\frac{Z_1}{2Z_2} + 1\right)e^{-\alpha} + 1 = 0$$

$$\frac{e^{-2\alpha} + 1}{2e^{-\alpha}} = \left(\frac{Z_1}{2Z_2} + 1\right)$$

$$\frac{e^{-\alpha} + e^{\alpha}}{2} = 1 + \frac{Z_1}{2Z_2}$$

$$\cosh\alpha = 1 + \frac{Z_1}{2Z_2}$$

Case-2: ***a*** **is a complex with modulus unity.**

So that we may write $a = e^{-j\beta}$. The wave is not attenuated at all, but suffers a change of phase by an angle β in each section, where,

$$e^{-j2\beta} - 2\left(\frac{Z_1}{2Z_2} + 1\right)e^{-j\beta} + 1 = 0$$

$$\frac{e^{-j2\beta} + 1}{2e^{-j\beta}} = \left(\frac{Z_1}{2Z_2} + 1\right)$$

$$\frac{e^{-j\beta} + e^{j\beta}}{2} = 1 + \frac{Z_1}{2Z_2}$$

$$\cos\beta = 1 + \frac{Z_1}{2Z_2}$$

Case-3: **a is then real and negative.**

So that the wave is attenuated with a phase change of π in successive sections. If we write $a = -e^{-\alpha}$. equation (1) becomes

$$e^{-2\alpha} + 2\left(\frac{Z_1}{2Z_2} + 1\right)e^{-\alpha} + 1 = 0$$

$$\frac{e^{-2\alpha} + 1}{2e^{-\alpha}} = -\left(\frac{Z_1}{2Z_2} + 1\right)$$

$$-\frac{e^{-\alpha} + e^{\alpha}}{2} = 1 + \frac{Z_1}{2Z_2}$$

$$-\cosh\alpha = 1 + \frac{Z_1}{2Z_2}$$

Characteristic Impedance and Propagation Constant of *T* Sections

The iterative impedance is the value of the impedance measured at one pair of terminals of the network when the other pair of terminals is terminated with an impedance of the same value. For example, the impedance looking into input terminals is 100 Ω when output terminals arc terminated at 100 Ω and the impedance looking into output terminals is 200 Ω when input terminals are terminated at 200 Ω. So, every asymmetrical four-terminal network has two iterative impedances. In a symmetrical network, these two impedances are equal. The common value is known as characteristic impedance.

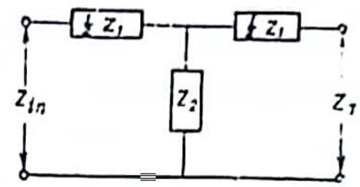


Fig. 1

The symmetrical T network is shown in fig. (1). The series arm consists of an impedance Z_1 and the shunt arm an impedance Z_2. If the output terminals of T section are closed through an impedance Z_T. the impedance across the input terminals is,

$$Z_{in} = \frac{Z_1}{2} + \frac{Z_2\left(\frac{Z_1}{2} + Z_T\right)}{Z_2 + \frac{Z_1}{2} + Z_T} \qquad (1)$$

If $Z_{in} = Z_T = Z_k$ then,

$$Z_k = \frac{Z_1}{2} + \frac{Z_2\left(\frac{Z_1}{2} + Z_k\right)}{Z_2 + \frac{Z_1}{2} + Z_k}$$

$$Z_k\left(Z_2 + \frac{Z_1}{2} + Z_k\right) = \frac{Z_1}{2}\left(Z_2 + \frac{Z_1}{2} + Z_k\right) + Z_2\left(\frac{Z_1}{2} + Z_k\right)$$

$$Z_k Z_2 + \frac{Z_k Z_1}{2} + {Z_k}^2 = \frac{Z_1 Z_2}{2} + \frac{{Z_1}^2}{4} + \frac{Z_1 Z_k}{2} + \frac{Z_2 Z_1}{2} + Z_2 Z_k$$

$${Z_k}^2 = Z_1 Z_2 + \frac{{Z_1}^2}{4}$$

$$Z_k = \sqrt{Z_1 Z_2 + \frac{{Z_1}^2}{4}} \qquad (2)$$

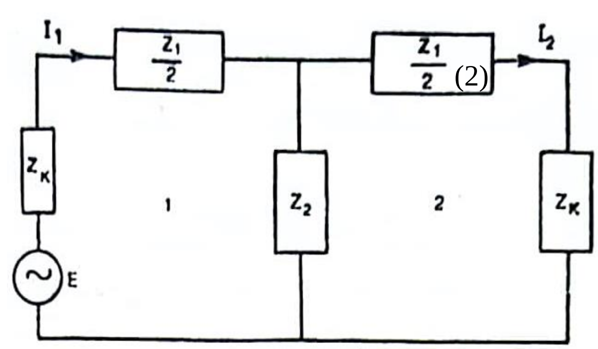


Fig. 2

Z_k is termed as the characteristic or iterative impedance of T-section. If the values of Z_1 and Z_2 are known, the value of Z_k can be calculated. To find out an expression for the propagation constant, we terminate the T section by its characteristic impedance Z_k. Now the input impedance Z_{in} will also be equal to Z_k. So, if we connect the generator of e.m.f. E with internal impedance Z_k at the input and terminate the output with Z_k, the section will be properly matched. This gives the maximum power output to the load. As shown in fig. 2, let the input and output currents be I_1 and I_2 respectively. Applying Kirchhoff's second law to mesh 2, we get

$$I_2 \frac{Z_1}{2} + I_2 Z_k - (I_1 - I_2) Z_2 = 0$$

$$I_2 \frac{Z_1}{2} + I_2 Z_k = I_1 Z_2 - I_2 Z_2$$

$$\frac{I_1}{I_2} = \frac{\frac{Z_1}{2} + Z_k + Z_2}{Z_2}$$

We know that $\frac{I_1}{I_2} = e^{\gamma}$ where γ is known as propagation constant. Hence,

$$e^{\gamma} = \frac{\frac{Z_1}{2} + Z_k + Z_2}{Z_2}$$

$$e^{\gamma} = 1 + \frac{Z_1}{2Z_2} + \frac{Z_k}{Z_2} \qquad (3)$$

$$\gamma = ln\left(1 + \frac{Z_1}{2Z_2} + \frac{Z_k}{Z_2}\right)$$

Also

$$e^{-\gamma} = \frac{1}{1 + \frac{Z_1}{2Z_2} + \frac{Z_k}{Z_2}}$$

$$= \left\{\left(1 + \frac{Z_1}{2Z_2}\right) + \frac{Z_k}{Z_2}\right\}^{-1}$$

$$= \left(1 + \frac{Z_1}{2Z_2}\right) - \frac{Z_k}{Z_2} + \cdots ....$$

$$e^{-\gamma} \approx 1 + \frac{Z_1}{2Z_2} - \frac{Z_k}{Z_2} \qquad (4)$$

From eqs. (3) and (4), we get

$$cosh\,\gamma = \frac{e^{\gamma} + e^{-\gamma}}{2} = \frac{1 + \frac{Z_1}{2Z_2} + \frac{Z_k}{Z_2} + 1 + \frac{Z_1}{2Z_2} - \frac{Z_k}{Z_2}}{2}$$

$$cosh\,\gamma = 1 + \frac{Z_1}{2Z_2} \qquad (5)$$

And

$$sinh\,\gamma = \frac{e^{\gamma} - e^{-\gamma}}{2} = \frac{1 + \frac{Z_1}{2Z_2} + \frac{Z_k}{Z_2} - 1 - \frac{Z_1}{2Z_2} + \frac{Z_k}{Z_2}}{2} = \frac{Z_k}{Z_2}$$

$$sinh\,\gamma = \frac{Z_k}{Z_2} \qquad (6)$$

Thus if Z_k and γ are given, the values of Z_1 and Z_2 can be calculated with the help of eqs. (5) and (6).

Characteristic Impedance and Propagation Constant of π Sections

The π network is shown in fig.1. If the output terminals of the π section are closed through an impedance Z_π, the impedance across the input terminals is

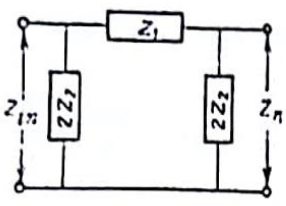


Fig. 1

$$Z_{in} = \frac{2Z_2\left(Z_1 + \frac{2Z_2Z_\pi}{2Z_2 + Z_\pi}\right)}{2Z_2 + Z_1 + \frac{2Z_2Z_\pi}{2Z_2 + Z_\pi}} \qquad (1)$$

If $Z_{in} = Z_\pi = Z'_k$

$$Z'_k = \frac{2Z_2\left(Z_1 + \frac{2Z_2Z'_k}{2Z_2 + Z'_k}\right)}{2Z_2 + Z_1 + \frac{2Z_2Z'_k}{2Z_2 + Z'_k}}$$

$$Z'_k 2Z_2 + Z'_k Z_1 + \frac{2Z_2{Z'_k}^2}{2Z_2 + Z'_k} = 2Z_2Z_1 + \frac{4{Z_2}^2Z'_k}{2Z_2 + Z'_k}$$

$$Z'_k 2Z_2(2Z_2 + Z'_k) + Z'_k Z_1(2Z_2 + Z'_k) + 2Z_2{Z'_k}^2 = 2Z_2Z_1(2Z_2 + Z'_k) + 4{Z_2}^2Z'_k$$

$$4{Z_2}^2Z'_k + 2Z_2{Z'_k}^2 + 2Z_1Z_2Z'_k + Z_1{Z'_k}^2 + 2Z_2{Z'_k}^2 = 4Z_1{Z_2}^2 + 2Z_1Z_2Z'_k + 4{Z_2}^2Z'_k$$

$$Z_1{Z'_k}^2 + 4Z_2{Z'_k}^2 = 4Z_1{Z_2}^2$$

$$(Z_1 + 4Z_2){Z'_k}^2 = 4Z_1{Z_2}^2$$

$$Z'_k = \sqrt{\frac{4Z_1{Z_2}^2}{Z_1 + 4Z_2}}$$

$$Z'_k = \sqrt{\frac{Z_1Z_2}{1 + \frac{Z_1}{4Z_2}}}$$

Z'_k is called the iterative impedance of π section. In order to find out the propagation constant of π -section. we connect a generator of e.m.f. E with internal impedance Z'_k at input terminals and terminate the π -section by the characteristic impedance Z'_k as shown in fig. (2). Let I_1 and I_2 he the input and output currents. Applying Kirchhoff's second law to mesh 3. we get

$$I_2 Z'_k - (I - I_2)2Z_2 = 0$$

$$I_2 Z'_k - 2IZ_2 + 2I_2 Z_2 = 0$$

$$I_2 = \frac{2IZ_2}{Z'_k + 2Z_2} \qquad (3)$$

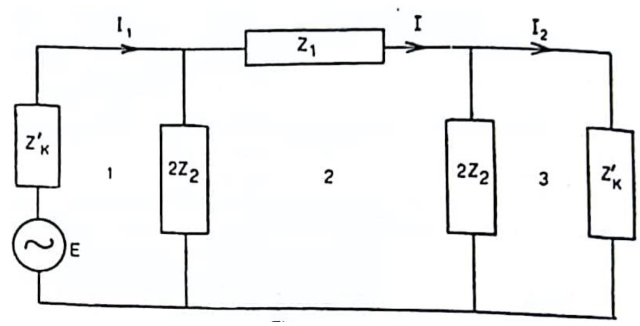


Fig. 2

Applying Kirchhoff's second law to mess 2, we have

$$IZ_1 + (I - I_2)2Z_2 - (I_1 - I)2Z_2 = 0$$

$$IZ_1 + 2IZ_2 - 2I_2 Z_2 - 2I_1 Z_2 + 2IZ_2 = 0$$

$$IZ_1 + 4IZ_2 - 2I_2 Z_2 - 2I_1 Z_2 = 0$$

Applying eqn.3 we get,

$$IZ_1 + 4IZ_2 - \frac{4I{Z_2}^2}{Z'_k + 2Z_2} - 2I_1 Z_2 = 0$$

$$\frac{I\left(Z_1 Z'_k + 2Z_1 Z_2 + 4Z_2 Z'_k + 8{Z_2}^2 - 4{Z_2}^2\right)}{Z'_k + 2Z_2} = 2I_1 Z_2$$

$$I_1 = \frac{I\left\{2Z_1 Z_2 + 4{Z_2}^2 + (Z_1 + 4Z_2)Z'_k\right\}}{2Z_2(Z'_k + 2Z_2)} \qquad (4)$$

From eqn. 3 and 4 we can write

$$\frac{I_1}{I_2} = \frac{\dfrac{I\left\{2Z_1 Z_2 + 4{Z_2}^2 + (Z_1 + 4Z_2)Z'_k\right\}}{2Z_2(Z'_k + 2Z_2)}}{\dfrac{2IZ_2}{Z'_k + 2Z_2}}$$

$$\frac{I_1}{I_2} = \frac{2Z_1 Z_2 + 4{Z_2}^2 + (Z_1 + 4Z_2)Z'_k}{4{Z_2}^2}$$

$$= 1 + \frac{Z_1}{2Z_2} + \frac{(Z_1 + 4Z_2)Z'_k}{4{Z_2}^2}$$

$$= 1 + \frac{Z_1}{2Z_2} + \frac{(Z_1 + 4Z_2)}{4{Z_2}^2}\sqrt{\frac{4Z_1{Z_2}^2}{Z_1 + 4Z_2}}$$

$$= 1 + \frac{Z_1}{2Z_2} + \frac{1}{Z_2}\sqrt{\frac{(Z_1 + 4Z_2)^2 4Z_1{Z_2}^2}{(4Z_2)^2(Z_1 + 4Z_2)}}$$

$$= 1 + \frac{Z_1}{2Z_2} + \frac{1}{Z_2}\sqrt{\frac{(Z_1 + 4Z_2)Z_1}{4}}$$

Md. Saifur Rahman, Lecturer, Department of Physics, University of Rajshahi / 3H / Chapter-2 / 2020

$$= 1 + \frac{Z_1}{2Z_2} + \frac{1}{Z_2}\sqrt{Z_1 Z_2 + \frac{{Z_1}^2}{4}}$$

$$= 1 + \frac{Z_1}{2Z_2} + \frac{Z_k}{Z_2}$$

$$\therefore \qquad e^{\gamma_\pi} = e^{\gamma_T} \tag{5}$$

Hence the propagation constant of π -section is the same as that of a T-section.

Filter Fundamentals: Pass and Stop Bands

Ideally, it is desired that a filter network transmits or passes the desired frequency band without loss, whereas it should stop or completely attenuate all undesired frequencies. The propagation constant $\gamma = \alpha + j\beta$, being a function of frequency, can supply information on the ability of the filter to perform as desired. If $\alpha = 0$ or $I_1 = I_2$ then there is no attenuation, only a phase shift, in transmitting a signal through the filter, and operation is in a **pass band** of frequencies. When α has a positive value, then I_2 is smaller in magnitude than I_1, attenuation has occurred and operation is in **attenuation** or **stop band** of frequencies.

We know the formula

$$sinh\frac{\gamma}{2} = \sqrt{\frac{1}{2}(cosh\,\gamma - 1)}$$

Using the value of $cosh\,\gamma$ previous section, $cosh\,\gamma = 1 + \frac{Z_1}{2Z_2}$, we can write

$$sinh\frac{\gamma}{2} = \sqrt{\frac{1}{2}\left(1 + \frac{Z_1}{2Z_2} - 1\right)}$$

$$sinh\frac{\gamma}{2} = \sqrt{\frac{Z_1}{4Z_2}}$$

It will first be assumed that the network contains only pure reactances, and thus $\frac{Z_1}{4Z_2}$ will be real, and either positive or negative, depending on the type of reactance used for Z_1 and Z_2. Expanding gives

$$sinh\frac{\alpha + j\beta}{2} = sinh\left(\frac{\alpha}{2} + \frac{j\beta}{2}\right) = \sqrt{\frac{Z_1}{4Z_2}}$$

$$sinh\left(\frac{\alpha}{2}\right)cosh\left(\frac{j\beta}{2}\right) + cosh\left(\frac{\alpha}{2}\right)sinh\left(\frac{j\beta}{2}\right) = \sqrt{\frac{Z_1}{4Z_2}}$$

$$\boldsymbol{sinh\left(\frac{\alpha}{2}\right)cos\left(\frac{\beta}{2}\right) + j\,cosh\left(\frac{\alpha}{2}\right)sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{Z_1}{4Z_2}}} \qquad (1)$$

as an equation containing much information.

If Z_1 and Z_2 are the same type of reactance then $\frac{Z_1}{4Z_2} > 0$, or the ratio $\frac{Z_1}{4Z_2}$ is positive and real. This requires that $sinh\frac{\gamma}{2}$ be real, which means that the imaginary term in Eq. 1 must equal zero and that

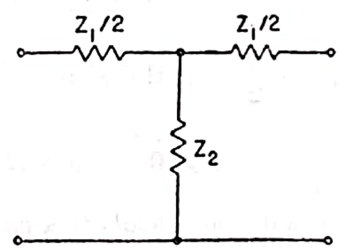
$$\text{(a) } cosh\left(\frac{\alpha}{2}\right)sin\left(\frac{\beta}{2}\right) = 0$$

$$\text{(b) } sinh\left(\frac{\alpha}{2}\right)cos\left(\frac{\beta}{2}\right) = \sqrt{\frac{Z_1}{4Z_2}}$$

are simultaneously satisfied. From (a), $sin\left(\frac{\beta}{2}\right) = 0$ $\quad [\because cosh^{-1}0 = Undefined\,]$

$$\frac{\beta}{2} = n\pi$$

where n = 0, 1, 2, . .



From (b), since $cos\left(\frac{\beta}{2}\right) = 1$, then $sinh\left(\frac{\alpha}{2}\right) = \sqrt{\frac{Z_1}{4Z_2}}$

and the attenuation will be given by

$$\alpha = 2\,sinh^{-1}\sqrt{\frac{Z_1}{4Z_2}} \qquad (2)$$

<u>Thus, the condition that</u> $\frac{Z_1}{4Z_2} > 0$ <u>implies a stop or attenuation band of frequencies.</u>

If Z_1 and Z_2 are opposite types of reactance then $\frac{Z_1}{4Z_2}$ is negative, $\frac{Z_1}{4Z_2} < 0$, and the radical of Eq. 1 is imaginary. The real term in Eq. 1 must then be zero, so that

$$\text{(c) } sinh\left(\frac{\alpha}{2}\right)cos\left(\frac{\beta}{2}\right) = 0$$

$$\text{(d) } cosh\left(\frac{\alpha}{2}\right)sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{Z_1}{4Z_2}}$$

must be satisfied. **Two** conditions are possible from the above:

i. $sinh\left(\frac{\alpha}{2}\right) = 0;\quad \alpha = 0;\quad$ therefore $cosh\left(\frac{\alpha}{2}\right) = 1$

So, from (d), $sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{Z_1}{4Z_2}}$

This condition leads to a pass band, or region of zero attenuation, which is limited by the upper limit on the sine, or by $sin\left(\frac{\beta}{2}\right) = 1$, or it is required that

$$-1 < \frac{Z_1}{4Z_2} < 0 \qquad \left[\because \textit{we are in } \frac{Z_1}{4Z_2} < 0 \textit{ region}\right]$$

The phase angle in this pass band will be given by

$$\beta = 2\, sin^{-1}\sqrt{\frac{Z_1}{4Z_2}} \tag{3}$$

ii. $cos\left(\frac{\beta}{2}\right) = 0;$ $\beta = (2n-1)\pi;$ therefore $sin\left(\frac{\beta}{2}\right) = \pm 1$

So, from (d), $cosh\left(\frac{\alpha}{2}\right) = \sqrt{\frac{Z_1}{4Z_2}}$

This condition leads to a stop or attenuation band- since $\alpha \neq 0$. The phase angle is π, and the attenuation is given by

$$\alpha = 2\, cosh^{-1}\sqrt{\frac{Z_1}{4Z_2}} \tag{4}$$

Because the hyperbolic cosine has no value below unity, it appears that the region in which condition **ii** applies is a stop band where

$$\frac{Z_1}{4Z_2} < -1$$

Values of $\frac{Z_1}{4Z_2}$ can then be classified into three regions, with corresponding values of α and β, these regions being bounded by $\frac{Z_1}{4Z_2}$ values of $+\infty, 0, -1$ and $-\infty$ as given below:

$\frac{Z_1}{4Z_2} =$	$+\infty$ **to** 0	0 **to** -1	-1 **to** $-\infty$
Reactance type	Same	Opposite	Opposite
Band	Stop	Pass	Stop
α	$2\, sinh^{-1}\sqrt{\frac{Z_1}{4Z_2}}$	0	$2\, cosh^{-1}\sqrt{\frac{Z_1}{4Z_2}}$
β	π	$2\, sin^{-1}\sqrt{\frac{Z_1}{4Z_2}}$	π

The frequencies at which the network changes from a pass network to a stop network, or vice versa, are called **cutoff frequencies**. These frequencies occur when

$$\frac{\boldsymbol{Z_1}}{\boldsymbol{4Z_2}} = \boldsymbol{0} \qquad \boldsymbol{or} \qquad \boldsymbol{Z_1 = 0} \tag{5}$$

$$\frac{\boldsymbol{Z_1}}{\boldsymbol{4Z_2}} = \boldsymbol{-1} \qquad \boldsymbol{or} \qquad \boldsymbol{Z_1 = -4Z_2} \tag{6}$$

where Z_1 and Z_2 are opposite types of reactance. Since Z_1 and Z_2 may have several configurations, as L and C elements, or as parallel and series combinations, a variety of types of performance are possible. The elements considered above were assumed pure reactances, and design is ordinarily carried out on this basis. Measurements of actual performance are then made and adjustments are introduced into the design to compensate for deviation of the results from the ideal. In addition to minimizing the losses of physical elements, it is also necessary to reduce stray electric and magnetic couplings between elements to obtain more nearly the predicted performance.

The Constant-K Low-Pass Filter

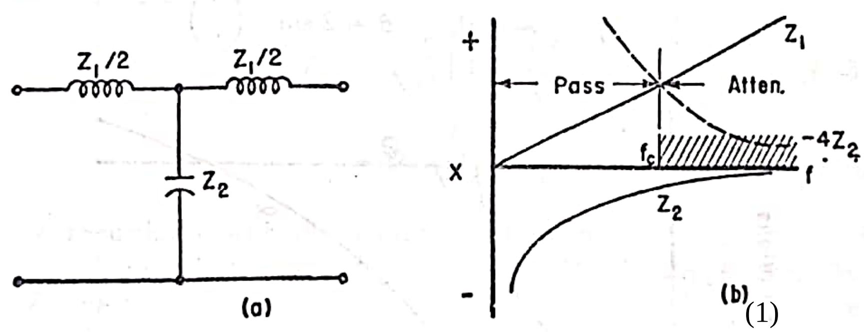
If Z_1 and Z_2 of a reactance network are unlike reactance arms, then

$$Z_1 Z_2 = k^2$$

Where k is a constant independent of frequency. Networks or filter sections for which this relation holds are called constant-k filters. As a special case, let $Z_1 = j\omega L$ and $Z_2 = \frac{1}{j\omega C} = \frac{-j}{\omega C}$, then the product

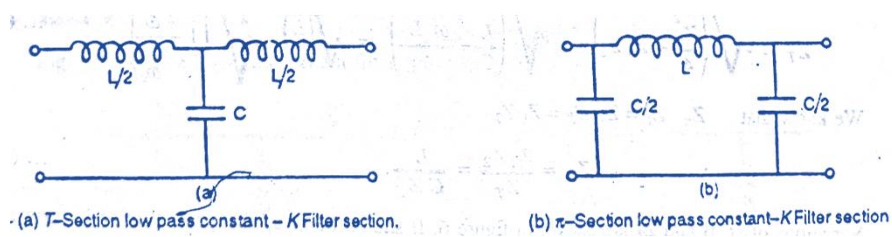
$$Z_1 Z_2 = j\omega L \frac{-j}{\omega C} = \frac{L}{C} = R_k^2$$

The term R_k is used since k must be real if Z_1 and Z_2 are of the opposite type. A T section so designed would appear as Fig. 1 (a).



(1)

Fig. 1: (a) Low-pass filter section; (b) reactance curves demonstrating that (a) is a low-pass section or has a pass band between $Z_1 = 0$ and $Z_1 = -4Z_2$.



The reactance of Z_1 and Z_2 will vary with frequency as sketched at (b), Fig. 1. The curve representing $-4Z_2$, maybe drawn and compared with the curve for Z_1. We know that a pass band starts at the frequency at which $Z_1 = 0$ and runs to the frequency at which $Z_1 = -4Z_2$. Thus, the reactance curves show that a pass band starts at $f = 0$ and continues to some higher frequency f_c. All frequencies above f_c lie in a stop, or attenuation, band. Thus, the network is called a low-pass filter. The cutoff frequency f_c maybe readily determined, since at that point

$$Z_1 = -4Z_2$$

$$j\omega_c L = 4\frac{j}{\omega_c C}$$

$$\omega_c = \sqrt{\frac{4}{LC}}$$

$$2\pi f_c = \sqrt{\frac{4}{LC}}$$

$$f_c = \frac{1}{\pi\sqrt{LC}} \qquad (2)$$

This expression may be used to develop certain relations applicable to the low-pass network. Then $sinh\frac{\gamma}{2}$ may be evaluated as

$$sinh\frac{\gamma}{2} = \sqrt{\frac{Z_1}{4Z_2}} = \sqrt{\frac{j\omega L}{4\frac{-j}{\omega C}}} = \sqrt{-\frac{\omega^2 LC}{4}} = j\frac{\omega\sqrt{LC}}{2} = j\frac{2\pi f\sqrt{LC}}{2}$$

and in view of Eq. 2 this is

$$sinh\frac{\gamma}{2} = j\frac{f}{f_c}$$

$$sinh\left(\frac{\alpha}{2}+\frac{j\beta}{2}\right) = j\frac{f}{f_c}$$

$$sinh\left(\frac{\alpha}{2}\right)cosh\left(\frac{j\beta}{2}\right) + cosh\left(\frac{\alpha}{2}\right)sinh\left(\frac{j\beta}{2}\right) = j\frac{f}{f_c}$$

$$sinh\left(\frac{\alpha}{2}\right)cos\left(\frac{\beta}{2}\right) + j\,cosh\left(\frac{\alpha}{2}\right)sin\left(\frac{\beta}{2}\right) = j\frac{f}{f_c}$$

$$\therefore \qquad sinh\left(\frac{\alpha}{2}\right)cos\left(\frac{\beta}{2}\right) = 0 \qquad (3)$$

$$and \qquad cosh\left(\frac{\alpha}{2}\right)sin\left(\frac{\beta}{2}\right) = \frac{f}{f_c} \qquad (4)$$

$$-1 < \frac{Z_1}{4Z_2} < 0$$

$$-1 < \left(j\frac{f}{f_c}\right)^2 < 0$$

$$-1 < -\left(\frac{f}{f_c}\right)^2 < 0$$

$$1 > \left(\frac{f}{f_c}\right)^2 > 0$$

$$1 > \frac{f}{f_c} > 0$$

Then if the frequency f is in the pass band, then $\alpha = 0$. So that $-1 < \frac{Z_1}{4Z_2} < 0$, i.e., $1 > \frac{f}{f_c} > 0$, then from eqn. (4)

$$sin\left(\frac{\beta}{2}\right) = \frac{f}{f_c}$$

$$\beta = 2\,sin^{-1}\left(\frac{f}{f_c}\right) \qquad (5)$$

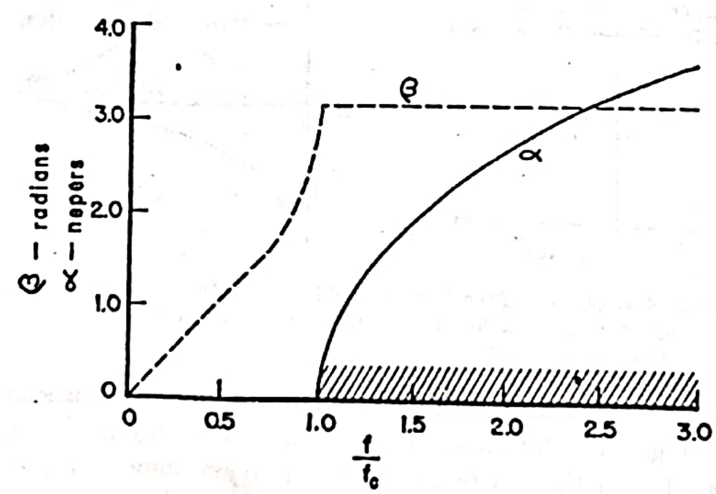


Fig. 2: Variation of α and β with frequency for the low-pass section

Whereas if frequency f is in the attenuation band, then $\alpha \neq 0$. So that $\frac{Z_1}{4Z_2} < -1$, or $\frac{f}{f_c} > 1$, then from eqn. (3)

$$cos\left(\frac{\beta}{2}\right) = 0; \quad \frac{\beta}{2} = \frac{\pi}{2}; \qquad \beta = \pi$$

and using this in eqn. (4)

$$cosh\left(\frac{\alpha}{2}\right) = \frac{f}{f_c}$$

$$\alpha = 2\, cosh^{-1}\left(\frac{f}{f_c}\right) \tag{6}$$

thereby allowing determination of α and β. The variation of α and β is plotted in Fig. 2 as a function of $\frac{f}{f_c}$. This method shows that the attenuation α is zero throughout the pass hand but rises gradually from the cutoff frequency at $\frac{f}{f_c} = 1$ to a value of ∞ at infinite frequency. The phase shift β is zero at zero frequency and increases gradually through the pass hand, reaching π at f_c and remaining at π for all higher frequencies.

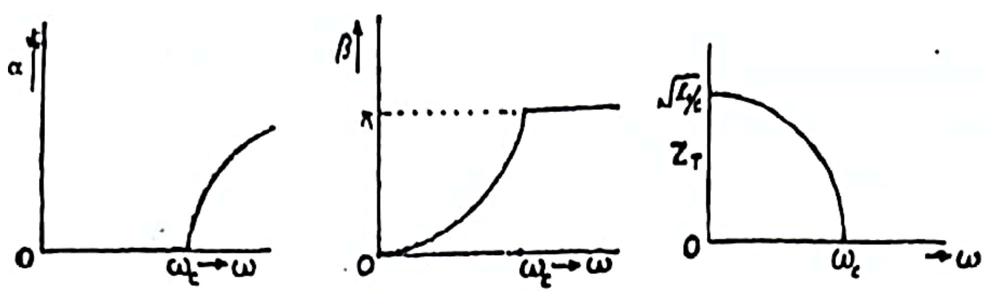


Fig. 3: Variation of α, β and real part of Z_T with frequency for the low-pass filter.

The characteristic impedance of a T section was obtained as

$$Z_T = \sqrt{Z_1 Z_2 + \frac{{Z_1}^2}{4}} = \sqrt{Z_1 Z_2 \left(1 + \frac{Z_1}{4Z_2}\right)}$$

$$Z_T = \sqrt{j\omega L \left(\frac{-j}{\omega C}\right) \left\{1 + \frac{j\omega L}{4\left(\frac{-j}{\omega C}\right)}\right\}}$$

$$Z_T = \sqrt{\frac{L}{C}\left(1 - \frac{\omega^2 LC}{4}\right)}$$

$$Z_T = \sqrt{\frac{L}{C}(1 - \pi^2 f^2 LC)}$$

$$Z_T = \sqrt{\frac{L}{C}\left(1 - \frac{f^2}{{f_c}^2}\right)} \qquad \left[Using\ eqn. 2\ \left({f_c}^2 = \frac{1}{\pi^2 LC}\right)\right]$$

$$Z_T = R_k \sqrt{1 - \left(\frac{f}{f_c}\right)^2}$$

It may be seen that Z_T varies throughout the pass band, reaching a value of zero at cutoff, then becomes imaginary in the attenuation band, rising to infinite reactance at infinite frequency. The characteristic impedance Z_T is real if $f < f_c$ and imaginary if $f > f_c$.

The Constant-K High-Pass Filter

If Z_1 and Z_2 of a reactance network are unlike reactance arms, then

$$Z_1 Z_2 = k^2$$

Where k is a constant independent of frequency. Networks or filter sections for which this relation holds are called constant-k filters. As a special case, let $Z_1 = \frac{1}{j\omega C} = \frac{-j}{\omega C}$ and $Z_2 = j\omega L$, then the product

$$Z_1 Z_2 = \frac{-j}{\omega C} j\omega L = \frac{L}{C} = R_k^2 \qquad (1)$$

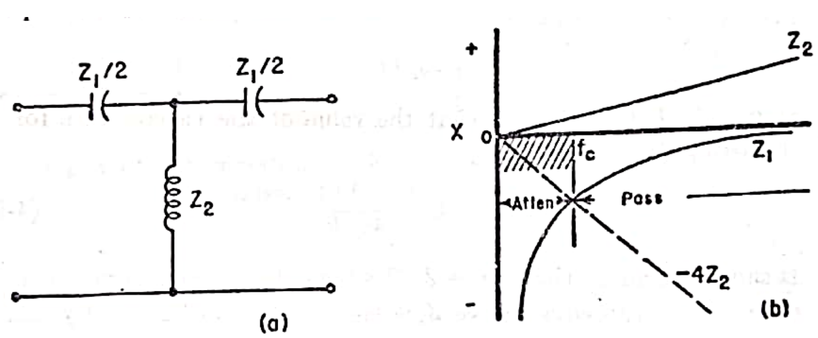
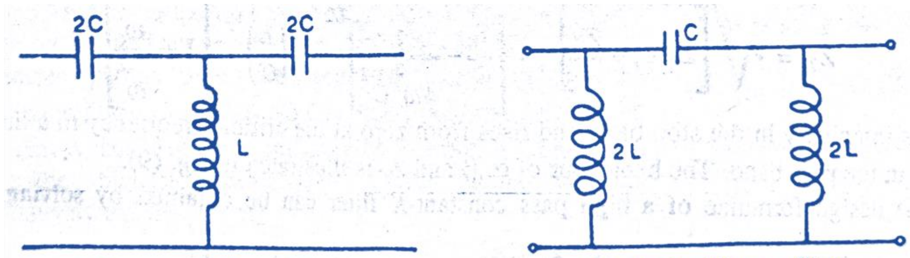


Fig.1: (a) High-pass filter section; (b) reactance curves demonstrating that (a) is a high-pass section or lists a pas band between $Z_1 = 0$ and $Z_1 = -4Z_2$

and the filter design obtained will be of the constant-k type. The T section will then appear as Fig. 1(a). The reactance of Z_1 and Z_2 are sketched as functions of frequency in Fig.1(b) and Z_1 is compared with $-4Z_2$ showing a cutoff frequency at the point at which $Z_1 = -4Z_2$, with a pass band from that frequency to infinity where $Z_1 = 0$. The network is thus a **high-pass** filter. All frequencies below f_c lie in an attenuation, or stop, band. The cutoff frequency is determined as the frequency at which $Z_1 = -4Z_2$



(a) T–section high pass filter (b) π–section high pass filter

$$\frac{j}{\omega_c C} = 4j\omega_c L$$

$$\omega_c = \sqrt{\frac{1}{4LC}}$$

$$2\pi f_c = \sqrt{\frac{1}{4LC}}$$

$$f_c = \frac{1}{4\pi\sqrt{LC}} \qquad (2)$$

This expression may be used to develop certain relations applicable to the low-pass network. Then $sinh\frac{\gamma}{2}$ may be evaluated as

$$sinh\frac{\gamma}{2} = \sqrt{\frac{Z_1}{4Z_2}} = \sqrt{\frac{\frac{-j}{\omega C}}{4j\omega L}} = \sqrt{-\frac{1}{4\omega^2 LC}} = j\frac{1}{2\omega\sqrt{LC}} = j\frac{1}{4\pi f\sqrt{LC}}$$

and in view of Eq. 2 this is

$$sinh\frac{\gamma}{2} = j\frac{f_c}{f}$$

$$sinh\left(\frac{\alpha}{2} + \frac{j\beta}{2}\right) = j\frac{f_c}{f}$$

$$sinh\left(\frac{\alpha}{2}\right)cosh\left(\frac{j\beta}{2}\right) + cosh\left(\frac{\alpha}{2}\right)sinh\left(\frac{j\beta}{2}\right) = j\frac{f_c}{f}$$

$$sinh\left(\frac{\alpha}{2}\right)cos\left(\frac{\beta}{2}\right) + j\,cosh\left(\frac{\alpha}{2}\right)sin\left(\frac{\beta}{2}\right) = j\frac{f_c}{f}$$

$$\therefore \quad sinh\left(\frac{\alpha}{2}\right)cos\left(\frac{\beta}{2}\right) = 0 \qquad (3)$$

$$and \quad cosh\left(\frac{\alpha}{2}\right)sin\left(\frac{\beta}{2}\right) = \frac{f_c}{f} \qquad (4)$$

$$-1 < \frac{Z_1}{4Z_2} < 0$$

$$-1 < \left(j\frac{f_c}{f}\right)^2 < 0$$

$$-1 < -\left(\frac{f_c}{f}\right)^2 < 0$$

$$1 > \left(\frac{f_c}{f}\right)^2 > 0$$

$$1 > \frac{f_c}{f} > 0$$

Then if the frequency f is in the pass band, then $\alpha = 0$. So that $-1 < \frac{Z_1}{4Z_2} < 0$, i.e., $1 > \frac{f_c}{f} > 0$, then from eqn. (4)

(5)

$$sin\left(\frac{\beta}{2}\right) = \frac{f_c}{f}$$

$$\beta = 2\, sin^{-1}\left(\frac{f_c}{f}\right)$$

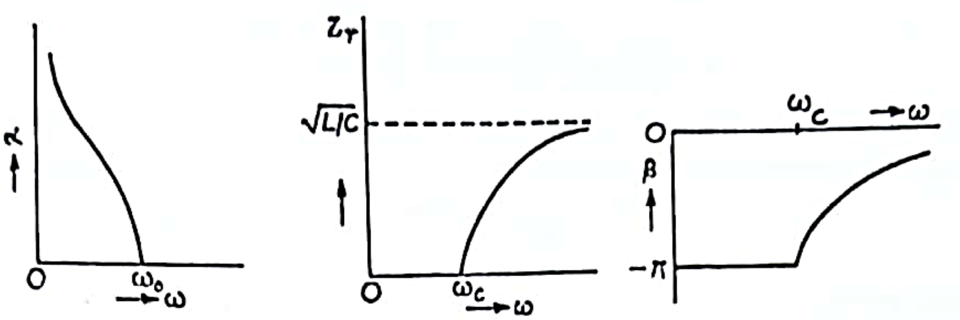
Whereas if frequency f is in the attenuation band, then $\alpha \neq 0$. So that $\frac{Z_1}{4Z_2} < -1$, or $\frac{f_c}{f} > 1$, then from eqn. (3)

$$cos\left(\frac{\beta}{2}\right) = 0; \quad \frac{\beta}{2} = \frac{\pi}{2}; \qquad \beta = \pi$$

and using this in eqn. (4)

$$cosh\left(\frac{\alpha}{2}\right) = \frac{f_c}{f}$$

$$\alpha = 2\, cosh^{-1}\left(\frac{f_c}{f}\right) \tag{6}$$



The region in which $\frac{f_c}{f} < 1$ is a pass band, so that the variation of γ inside and outside the pass band will be identical with the values for the low-pass filter, and the curves of Fig. 2 of the previous section will apply if the abscissa be considered as calibrated in terms of $\frac{f_c}{f}$, except that the phase angle β will be negative, changing from 0 at the infinite frequency or $\frac{f_c}{f} = 0$, to $-\pi$ at cutoff or $\frac{f_c}{f} = 1$.

The characteristic impedance of a T section was obtained as

$$Z_T = \sqrt{Z_1 Z_2 + \frac{{Z_1}^2}{4}} = \sqrt{Z_1 Z_2\left(1 + \frac{Z_1}{4Z_2}\right)}$$

$$Z_T = \sqrt{\left(\frac{-j}{\omega C}\right) j\omega L \left\{1 + \frac{\left(\frac{-j}{\omega C}\right)}{4j\omega L}\right\}}$$

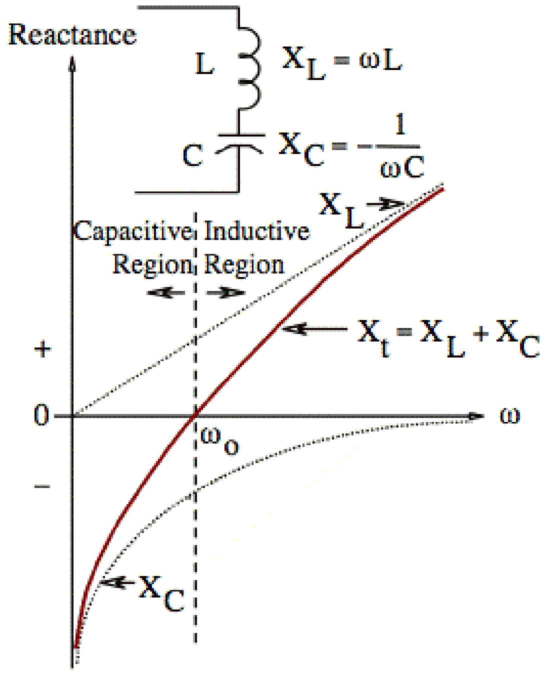
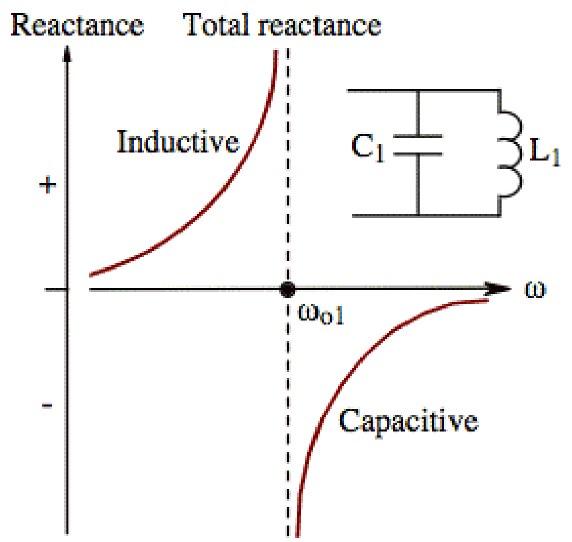
$$Z_T = \sqrt{\frac{L}{C}\left(1 - \frac{1}{4\omega^2 LC}\right)}$$

$$Z_T = \sqrt{\frac{L}{C}\left(1 - \frac{1}{16\pi^2 f^2 LC}\right)}$$

$$Z_T = \sqrt{\frac{L}{C}\left(1 - \frac{{f_c}^2}{f^2}\right)} \qquad \left[Using\ eqn.2\ \left({f_c}^2 = \frac{1}{16\pi^2 LC}\right)\right]$$

$$Z_T = R_k\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

It may be seen that Z_T varies throughout the pass band, reaching a value of zero at cutoff, then becomes imaginary in the attenuation band, rising to infinite reactance at infinite frequency. The characteristic impedance Z_T is real if $f > f_c$ and imaginary if $f < f_c$.

$X_{Series} = \omega L - \frac{1}{\omega C}$	$X_{Parallel} = \frac{\omega L\left(-\frac{1}{\omega C}\right)}{\omega L + \left(-\frac{1}{\omega C}\right)} = \frac{-\frac{L}{C}}{\omega L - \frac{1}{\omega C}}$
	Let, $L = 2H$, $\omega = 4Hz$, $C = \frac{1}{32}F$; $\left[\omega L = \frac{1}{\omega C}\right]$ 

ω	1	2	3	4	5	6	7
X	2.1	5.3	13.7	∞	-17.8	-9.6	-6.8

Band Pass Filter

A band pass filter transmits a certain range of frequencies and attenuates all others. Occasionally it is desirable to pass a band of frequencies and to attenuate frequencies on both sides of the pass band. The action might be thought of as that of low-pass and high-pass filters in **series**, in which the cutoff frequency of the low-pass filter is above the cutoff frequency of the high-pass filter, the overlap thus allowing only a band of frequencies to pass. Although such a design would function, it is more economical to combine the low- and high-pass functions into a single filter section.

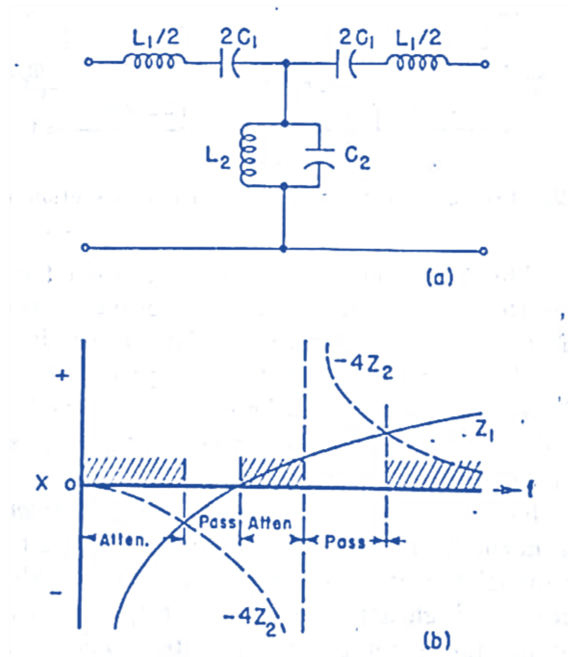


Fig. 1: Band-pass filter network; (b) reactance curves showing possibility of two bands.

Consider the circuit of Fig. 1(a), with a series-resonant series arm and an antiresonant shunt arm. In general, the reactance curves show that two pass bands might exist. If, however, the antiresonant frequency of the shunt arm is made to correspond to the resonant frequency of the series arm, the reactance curves become as shown in Fig. 2 and only one pass band appears. For this condition of equal resonant frequencies,

$$\omega_0 L_1 = \frac{1}{\omega_0 C_1}$$

$$\omega_0 L_2 = \frac{1}{\omega_0 C_2}$$

$$\omega_0{}^2 L_1 C_1 = 1$$

$$\omega_0{}^2 L_2 C_2 = 1$$

$$\omega_0{}^2 L_1 C_1 = 1 = \omega_0{}^2 L_2 C_2$$

$$L_1 C_1 = L_2 C_2 \qquad (1)$$

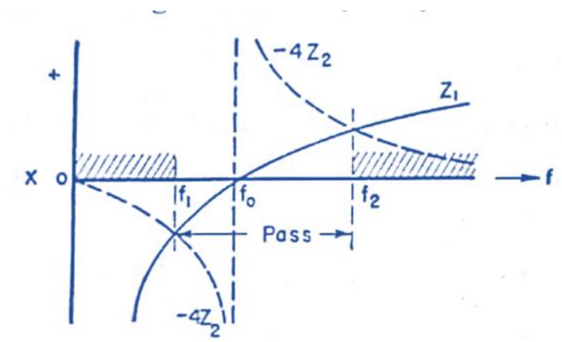


Fig. 2: Reactance curves for the band-pass network when resonant and antiresonant frequencies are properly adjusted.

The impedances of the arms are

$$Z_1 = j\omega L_1 - \frac{j}{\omega C_1} = j\left(\frac{\omega^2 L_1 C_1 - 1}{\omega C_1}\right) \qquad (2)$$

$$Z_2 = \frac{j\omega L_2\left(-\frac{j}{\omega C_2}\right)}{j\omega L_2 + \left(-\frac{j}{\omega C_2}\right)} = \left(\frac{\frac{L_2}{C_2}}{j\left(\omega L_2 - \frac{1}{\omega C_2}\right)}\right) = -j\left(\frac{\omega L_2}{\omega^2 L_2 C_2 - 1}\right) \qquad (3)$$

That a network such as, Fig. 1(a) is still a constant-k filter is easily shown as

$$Z_1 Z_2 = -j\left(\frac{\omega^2 L_1 C_1 - 1}{\omega C_1}\right) j\left(\frac{\omega L_2}{\omega^2 L_2 C_2 - 1}\right)$$

$$= \left(\frac{\omega^2 L_1 C_1 - 1}{\omega C_1}\right)\left(\frac{\omega L_2}{\omega^2 L_2 C_2 - 1}\right) \qquad \left[\frac{1}{j\omega C_2} = \frac{j}{j^2 \omega C_2}\right]$$

And if $L_1C_1 = L_2C_2$, then from above equation

$$Z_1Z_2 = \frac{L_2}{C_1} = \frac{L_1}{C_2} = {R_k}^2 \tag{4}$$

Now we calculate

$$\frac{Z_1}{Z_2} = \frac{j\left(\frac{\omega^2 L_1 C_1 - 1}{\omega C_1}\right)}{-j\left(\frac{\omega L_2}{\omega^2 L_2 C_2 - 1}\right)}$$

$$= -\frac{(\omega^2 L_1 C_1 - 1)(\omega^2 L_2 C_2 - 1)}{\omega^2 L_2 C_1}$$

$$= -\frac{\left(\frac{\omega^2}{{\omega_0}^2} - 1\right)\left(\frac{\omega^2}{{\omega_0}^2} - 1\right)}{\omega^2 L_2 C_1} \qquad \left[\because {\omega_0}^2 = \frac{1}{L_1C_1} = \frac{1}{L_2C_2}\right]$$

$$= -\frac{\left(\frac{\omega^2}{{\omega_0}^2} - 1\right)^2}{\omega^2 L_2 C_1} \tag{5}$$

Thus, the previously developed theory still applies. At the cutoff frequencies,

$$\frac{Z_1}{4Z_2} = 0$$

And

$$\frac{Z_1}{4Z_2} = -1$$

If we consider $\frac{Z_1}{4Z_2} = 0$, then $Z_1 = j\omega L_1 - \frac{j}{\omega C_1} = 0$, which is only possible at the resonance frequency. So, $\omega = \omega_0$, which cannot be critical frequency.

If we consider $\frac{Z_1}{4Z_2} = -1$, then,

$$\frac{1}{4}\frac{\left(\frac{\omega^2}{{\omega_0}^2} - 1\right)^2}{\omega^2 L_2 C_1} = 1$$

$$\frac{\omega^4}{{\omega_0}^4} - 2\frac{\omega^2}{{\omega_0}^2} + 1 = 4\omega^2 L_2 C_1$$

$$\frac{\omega^4}{{\omega_0}^4} - 2\frac{\omega^2}{{\omega_0}^2} + 1 = \frac{4\omega^2 L_2 C_2 C_1}{C_2}$$

$$\frac{\omega^4}{{\omega_0}^4} - 2\frac{\omega^2}{{\omega_0}^2} + 1 = \frac{4\omega^2 C_1}{{\omega_0}^2 C_2}$$

$$\omega^4 - 2\omega^2{\omega_0}^2\left(1 + \frac{2C_1}{C_2}\right) + {\omega_0}^4 = 0 \tag{6}$$

The equation gives the critical frequencies:

$$\omega^2 = \frac{2\omega_0{}^2\left(1+\frac{2C_1}{C_2}\right) \pm \sqrt{4\omega_0{}^4\left(1+\frac{2C_1}{C_2}\right)^2 - 4\omega_0{}^4}}{2}$$

$$\omega = \omega_0\sqrt{\left(1+\frac{2C_1}{C_2}\right) \pm \sqrt{\left(1+\frac{2C_1}{C_2}\right)^2 - 1}}$$

If the roots of this equation is ω_1 and ω_2 then

$$\omega_1 = \omega_0\sqrt{\left(1+\frac{2C_1}{C_2}\right) + \sqrt{\left(1+\frac{2C_1}{C_2}\right)^2 - 1}} \quad (7)$$

$$\omega_2 = \omega_0\sqrt{\left(1+\frac{2C_1}{C_2}\right) - \sqrt{\left(1+\frac{2C_1}{C_2}\right)^2 - 1}} \quad (8)$$

Multiplying the above equations, we get

$$\omega_1\omega_2 = \omega_0{}^2\sqrt{\left\{\left(1+\frac{2C_1}{C_2}\right) + \sqrt{\left(1+\frac{2C_1}{C_2}\right)^2 - 1}\right\}\left\{\left(1+\frac{2C_1}{C_2}\right) - \sqrt{\left(1+\frac{2C_1}{C_2}\right)^2 - 1}\right\}}$$

$$\omega_1\omega_2 = \omega_0{}^2\sqrt{\left(1+\frac{2C_1}{C_2}\right)^2 - \left\{\left(1+\frac{2C_1}{C_2}\right)^2 - 1\right\}}$$

$$\omega_1\omega_2 = \omega_0{}^2$$

$$\omega_0 = \sqrt{\omega_1\omega_2}$$

Hence, the resonance frequency is the geometric mean of two cut off frequencies. The cut off angular frequencies are given by equations (7) and (8).

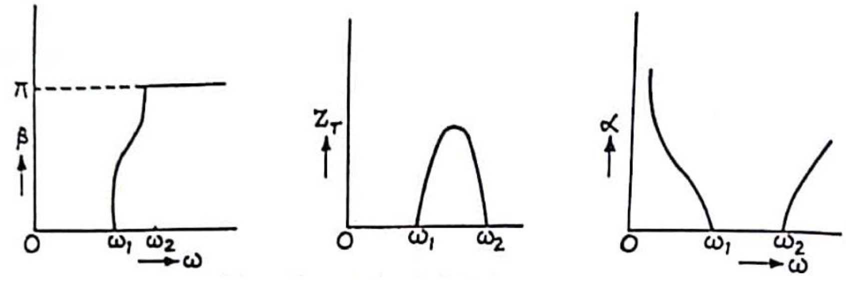


Fig. 3: Representing the behavior of β, real part of Z_T and α

Band-elimination filters

If the series- and parallel-tuned arms of the band-pass filter are interchanged, the result is the band-elimination filter shown at fig.1 (a). That this circuit does eliminate or attenuate a given frequency band is shown by the reactance curves for Z_1 and $-4Z_2$ at fig.1 (b). The action may be thought of as that of a low-pass filter in **parallel** with a high-paw; section, in which the, cut-off frequency of the low-pass filter is below that of the high-pass filter,

As for the band-pass filter, the series and shunt arms are made antiresonant and resonant at the same frequency f_0.

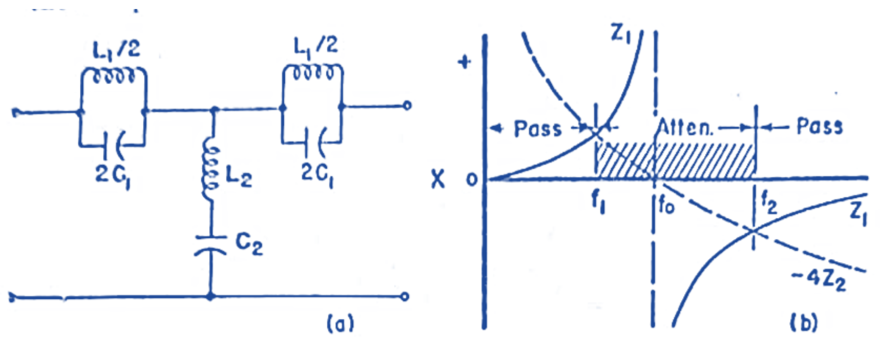


Fig. 1: (a) Band-elimination filter; (b) reactance curves showing action of band-elimination section.

If Z_1 represents the total series arm impedance and Z_2, total shunt arm impedance then,

$$Z_1 = \frac{j\omega L_1 \frac{1}{j\omega C_1}}{j\omega L_1 + \frac{1}{j\omega C_1}}$$

$$= \frac{\frac{L_1}{C_1}}{j\left(\frac{\omega^2 L_1 C_1 - 1}{\omega C_1}\right)}$$

$$Z_1 = \frac{\omega L_1}{j(\omega^2 L_1 C_1 - 1)} \qquad (1)$$

$$Z_2 = j\omega L_2 + \frac{1}{j\omega C_2}$$

$$Z_2 = j\left(\frac{\omega^2 L_2 C_2 - 1}{\omega C_2}\right) \qquad (2)$$

Now,

$$Z_1 Z_2 = \frac{\omega L_1}{j(\omega^2 L_1 C_1 - 1)} j\left(\frac{\omega^2 L_2 C_2 - 1}{\omega C_2}\right)$$

$$= \frac{L_1}{C_2} = \frac{L_2}{C_1} = {R_k}^2 \qquad (3)$$

$$[\because\ L_1 C_1 = L_2 C_2]$$

The arrangement in this type of filter is also same as in band pass filter, i.e., constant k filter.

Now we calculate

$$\frac{Z_1}{Z_2}=\frac{\dfrac{\omega L_1}{j(\omega^2 L_1 C_1-1)}}{j\left(\dfrac{\omega^2 L_2 C_2-1}{\omega C_2}\right)}$$

$$=-\frac{\dfrac{\omega L_1}{(\omega^2 L_1 C_1-1)}}{\left(\dfrac{\omega^2 L_2 C_2-1}{\omega C_2}\right)}$$

$$=-\frac{\dfrac{\omega^2 L_1 C_1 C_2}{C_1}}{(\omega^2 L_1 C_1-1)^2} \qquad [\because\ L_1 C_1 = L_2 C_2]$$

$$=-\frac{\dfrac{\omega^2 C_2}{{\omega_0}^2 C_1}}{\left(\dfrac{\omega^2}{{\omega_0}^2}-1\right)^2} \qquad \left[\because {\omega_0}^2=\frac{1}{L_1 C_1}=\frac{1}{L_2 C_2}\right] \qquad (4)$$

Thus, the previously developed theory still applies. At the cutoff frequencies,

$$\frac{Z_1}{4Z_2}=0$$

And

$$\frac{Z_1}{4Z_2}=-1$$

If we consider $\frac{Z_1}{4Z_2}=0$, then $Z_1=\frac{\omega L_1}{j(\omega^2 L_1 C_1-1)}=0$, which is only possible at the resonance frequency. So, $\omega=0$.

If we consider $\frac{Z_1}{4Z_2}=-1$, then,

$$\frac{1}{4}\frac{\dfrac{\omega^2 C_2}{{\omega_0}^2 C_1}}{\left(\dfrac{\omega^2}{{\omega_0}^2}-1\right)^2}=1$$

$$\frac{\omega^2 C_2}{{\omega_0}^2 C_1}=4\left(\frac{\omega^4}{{\omega_0}^4}-2\frac{\omega^2}{{\omega_0}^2}+1\right)$$

$$4\frac{\omega^4}{{\omega_0}^4}-8\frac{\omega^2}{{\omega_0}^2}+4-\frac{\omega^2 C_2}{{\omega_0}^2 C_1}=0$$

$$\omega^4-2\omega^2{\omega_0}^2+{\omega_0}^4-\frac{\omega^2{\omega_0}^2 C_2}{4C_1}=0$$

$$\omega^4 - 2\omega^2{\omega_0}^2\left(1+\frac{C_2}{8C_1}\right) + {\omega_0}^4 -= 0$$ (5)

The equation gives the critical frequencies:

$$\omega^2 = \frac{2{\omega_0}^2\left(1+\frac{C_2}{8C_1}\right) \pm \sqrt{4{\omega_0}^4\left(1+\frac{C_2}{8C_1}\right)^2 - 4{\omega_0}^4}}{2}$$

$$\omega = \omega_0\sqrt{\left(1+\frac{C_2}{8C_1}\right) \pm \sqrt{\left(1+\frac{C_2}{8C_1}\right)^2 - 1}}$$

If the roots of this equation are ω_1 and ω_2 then

$$\omega_1 = \omega_0\sqrt{\left(1+\frac{C_2}{8C_1}\right) - \sqrt{\left(1+\frac{C_2}{8C_1}\right)^2 - 1}} \qquad (6)$$

$$\omega_2 = \omega_0\sqrt{\left(1+\frac{C_2}{8C_1}\right) + \sqrt{\left(1+\frac{C_2}{8C_1}\right)^2 - 1}} \qquad (7)$$

Multiplying above equations, we get

$$\omega_1\omega_2 = {\omega_0}^2\sqrt{\left\{\left(1+\frac{C_2}{8C_1}\right) - \sqrt{\left(1+\frac{C_2}{8C_1}\right)^2 - 1}\right\}\left\{\left(1+\frac{C_2}{8C_1}\right) + \sqrt{\left(1+\frac{C_2}{8C_1}\right)^2 - 1}\right\}}$$

$$\omega_1\omega_2 = {\omega_0}^2\sqrt{\left(1+\frac{C_2}{8C_1}\right)^2 - \left\{\left(1+\frac{C_2}{8C_1}\right)^2 - 1\right\}}$$

$$\omega_1\omega_2 = {\omega_0}^2$$

$$\omega_0 = \sqrt{\omega_1\omega_2}$$

Hence, the resonance frequency is the geometric mean of two cut off frequencies. The cut off angular frequencies are given by equations (6) and (7). Thus, the frequencies lying between ω_1 *and* ω_2 are stopped while those lying outside this band are allowed to pass.

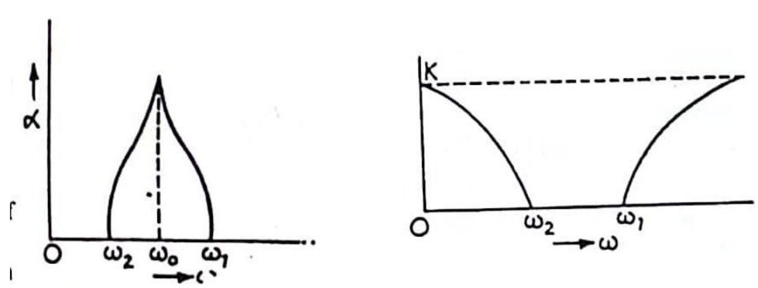


Fig. 2: Representing the behavior of real part of α and Z_T.

Filter Design

Low-Pass Filter Design

The design of a low-pass filter may be readily carried out. From the relation that at cutoff

$$Z_1 = -4Z_2$$

For low pass filter, Z_1 arm is inductive and Z_2 arm is capacitive. Then it is seen that

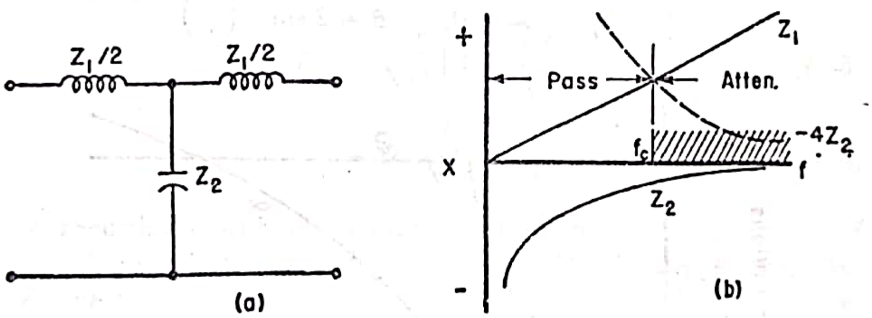


Fig. 1: (a) Low-pass filter section; (b) reactance curves demonstrating that (a) is a low-pass section or has a pass band between $Z_1 = 0$ and $Z_1 = -4Z_2$.

$$\omega_c L = \frac{4}{\omega_c C}$$

$$2\pi f_c L = \frac{4}{2\pi f_c C}$$

$$\pi^2 f_c^{\,2} LC = 1 \qquad (1)$$

Where f_c is the cutoff frequency. We know for constant k filters, $R_k^{\,2} = \frac{L}{C}$, which gives for the value of the arms. Inserting the value of C in equation 1 we get

$$\pi^2 f_c^{\,2} L \frac{L}{R_k^{\,2}} = 1 \qquad (2)$$

$$L = \frac{R_k}{\pi f_c}$$

By similar methods, the capacitance is obtained as

$$\pi^2 f_c^{\,2} C C R_k^{\,2} = 1$$

$$C = \frac{1}{\pi f_c R_k} \qquad (3)$$

Since the design is based on an impedance match at zero frequency only, power transfer to a matched load will drop at higher pass-band frequencies.

High-Pass Filter Design

The high-pass filter may be designed by again choosing a resistive load R equal to R_k such that

$$R_k^{\,2} = \frac{L}{C},$$

From the relation that at cutoff

$$Z_1 = -4Z_2$$

Md. Saifur Rahman, Lecturer, Department of Physics, University of Rajshahi / 3H / Chapter-2 / 2020

For low high pass filter, Z_1arm is capacitive and Z_2 arm is inductive. Then it is seen that

$$\frac{1}{\omega_c C} = 4\omega_c L$$

$$8\pi f_c L = \frac{1}{2\pi f_c C}$$

$$16\pi^2 {f_c}^2 LC = 1 \qquad (1)$$

Where f_c is the cutoff frequency. We know for constant k filters, ${R_k}^2 = \frac{L}{C}$, which gives for the value of the arms. Inserting the value of C in equation 1 we get

$$16\pi^2 {f_c}^2 L \frac{L}{{R_k}^2} = 1$$

$$L = \frac{R_k}{4\pi f_c} \qquad (2)$$

By similar methods, the capacitance is obtained as

$$16\pi^2 {f_c}^2 CC{R_k}^2 = 1$$

$$C = \frac{1}{4\pi f_c R_k} \qquad (3)$$

Problem 1: Design a low pass filter having a cut off frequency $f_c = 1000\ Hz$ and a design impedance of $500\ \Omega$.

Ans: Here

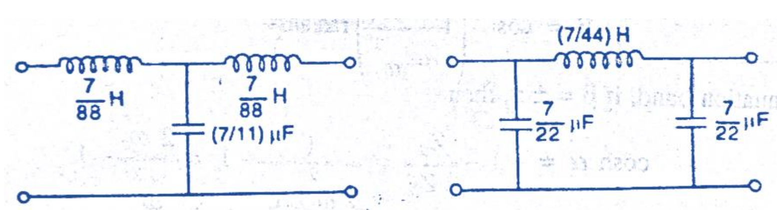
$R_k = \sqrt{\frac{L}{C}} = 500\ \Omega\ ,\ f_c = 1000\ Hz$

$$\therefore\ L = \frac{R_k}{\pi f_c} = \frac{500}{3.1416 \times 1000}\ H$$

$$= \frac{7}{44} H = 1.59\ H$$

$$And \quad C = \frac{1}{\pi f_c R_k} = \frac{1}{3.1416 \times 500\ \times 1000} = \frac{7}{11} \mu F$$

The T and π sections of this filter are shown in fig below:



Problem 2: Design a high pass filter to have a design impedance of $500\ \Omega$ and a cut off frequency $f_c = 1000\ Hz$.

Ans: Here

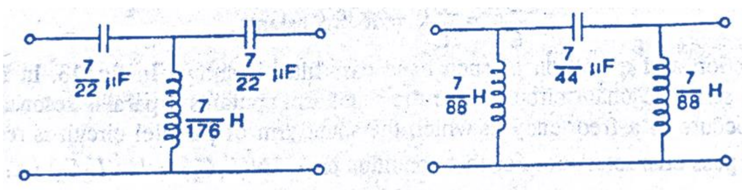
$R_k = \sqrt{\frac{L}{C}} = 500\ \Omega\ ,\ f_c = 1000\ Hz$

$$\therefore\ L = \frac{R_k}{4\pi f_c} = \frac{500}{4 \times 3.1416 \times 1000}\ H$$

$$= \frac{7}{176} H = 1.59\ H$$

$$And \quad C = \frac{1}{4\pi f_c R_k} = \frac{1}{4 \times 3.1416 \times 500\ \times 1000} = \frac{7}{44}\mu F$$

The T and π sections of this filter are shown in fig below:



Disadvantages of Simple Types of Filter

Filters considered earlier belong to the class of 'K derived' or 'constant K' filters since their impedances obey the relation

$$Z_1 Z_2 = K^2$$

Where K is a constant, independent of frequency. This constant K type filter suffers from two principal disadvantages:

(1) Its characteristic impedance is not sufficiently constant over the pass band but varies with frequency. The filter, therefore, can not be terminated correctly throughout the pass band.
(2) The attenuation does not rise very abruptly at the boundary to the transmission band. In order to overcome the inherent limitations of constant K-type, *Otto Zobel* devised a filter section which he called the 'm-derived' type filter. Such types of filters give practically uniform characteristic impedance over a large part of the pass band and at the same time increase the abruptness with which cut-off occurs.

A high degree of attenuation beyond the cut-off or a constant impedance in the pass band demands a more complicated structure. If the constant K section is regarded as the prototype, it is possible to design a section to have the same impedance and hence the same pass and attenuation hands but with a different degree of attenuation outside the pass band.

Introduction to m-Derived Filters (T—SECTION)

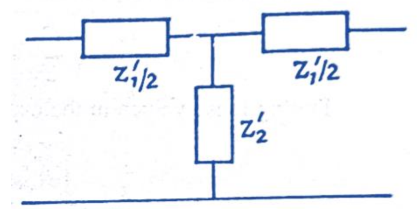


Fig. 1: Modified prototype T-section

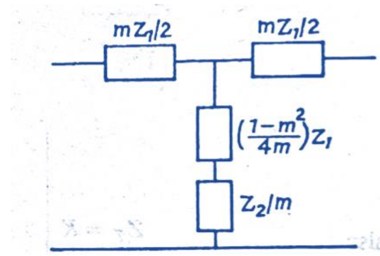


Fig. 2: m-derived T-section

Suppose that T-section, as shown in figure 1, has the series arm modified by some constant m. Then if this new section is to have the same impedance Z_T as the prototype. the shunt impedance must be modified in the same way. For the prototype section,

$$Z_T = \sqrt{Z_1 Z_2 \left(1 + \frac{Z_1}{4Z_2}\right)}$$

If for this new section $Z_1' = mZ_1$ and impedance $Z_T{}' = Z_T$, then

$$Z_T{}' = \sqrt{mZ_1 Z_2' \left(1 + \frac{mZ_1}{4Z_2'}\right)}$$

where Z_2' is the modified shunt impedance in the new section. If $Z_T{}' = Z_T$ then Z_2' must have the value determined by

$$Z_1 Z_2 \left(1 + \frac{Z_1}{4Z_2}\right) = mZ_1 Z_2' \left(1 + \frac{mZ_1}{4Z_2'}\right)$$

$$Z_1 Z_2 + \frac{Z_1{}^2}{4} = mZ_1 Z_2' + \frac{m^2 Z_1{}^2}{4}$$

$$mZ_1 Z_2' = Z_1 Z_2 + \frac{Z_1{}^2}{4}(1 - m^2)$$

$$Z_2' = \frac{Z_2}{m} + \frac{(1 - m^2)}{4m} Z_1 \tag{1}$$

and so the derived T section, as shown in fig. 2, has the same impedance as the prototype T section. m must always be chosen much that $0 < m < 1$. Now we shall discuss the pass band limit in this type of filter.

Pass band limit: For pass band in prototype T section, we know that $\frac{Z_1}{4Z_2} = 0 \; or -1$. Considering m derived T section, we write

$$\frac{Z_1'}{4Z_2'} = \frac{mZ_1}{4\left\{\frac{Z_2}{m} + \frac{(1 - m^2)}{4m} Z_1\right\}} = \frac{m\frac{Z_1}{Z_2}}{4\left\{\frac{1}{m} + \frac{1 - m^2}{4m} . \frac{Z_1}{Z_2}\right\}} \tag{1*}$$

If $\frac{Z_1}{4Z_2} = 0$, then $\frac{Z_1'}{4Z_2'} = 0$

Also if $\frac{Z_1}{4Z_2} = -1$ or $\frac{Z_1}{Z_2} = -4$ then

$$\frac{Z_1'}{4Z_2'} = \frac{-4m}{4\left\{\frac{1}{m} - \frac{1 - m^2}{m}\right\}} = -1$$

Thus pass hand in m derived T section will be determined by

$$\frac{Z_1'}{4Z_2'} = 0 \quad and \quad \frac{Z_1'}{4Z_2'} = -1 \tag{2}$$

Since the pass band limit in the case of 7n-derived T-section is the same as in the case of constant K type, the critical frequency of prototype and m-derived filter is the same. The shunt arm is to be chosen in such a way that if it is resonant at some frequency of infinite or high attenuation called f_∞ above f_c, This means that at the resonant frequency

$$\left|\frac{Z_2}{m}\right| = \left|\frac{(1-m^2)}{4m} Z_1\right|$$

and for low pass filter

$$\frac{1}{2\pi f_\infty Cm} = \frac{(1-m^2)}{4m} 2\pi f_\infty L$$

$$f_\infty = \frac{1}{\pi\sqrt{(1-m^2)LC}} \tag{3}$$

We know that the cut-off frequency for the low-pass filter is

$$f_c = \frac{1}{\pi\sqrt{LC}}$$

so that the Frequency of infinite attenuation will be

$$f_\infty = \frac{f_c}{\sqrt{(1-m^2)}} \tag{4}$$

from which

$$m = \sqrt{1 - \left(\frac{f_c}{f_\infty}\right)^2} \tag{5}$$

This equation determines the value of m to be used for a particular f_∞. Similar relations for the high pass filter can be derived; they are

$$f_\infty = f_c\sqrt{(1-m^2)} \tag{6}$$

$$m = \sqrt{1 - \left(\frac{f_\infty}{f_c}\right)^2} \tag{7}$$

The m derived section is designed by following the design of the prototype T section. The variation of attenuation over the attenuation band for the low pass m derived section in the stop band is dependent on the sign of reactances

$$\alpha = 2\cosh^{-1}\sqrt{\frac{Z_1}{4Z_2}} \qquad or \qquad \alpha = 2\sinh^{-1}\sqrt{\frac{Z_1}{4Z_2}}$$

$$f_c < f < f_\infty \qquad\qquad f_\infty < f$$

For $Z_1 = j\omega L$ and $Z_2 = -\frac{j}{\omega C}$ for the prototype, then from Eqn. 1*

$$\left|\frac{Z_1'}{4Z_2'}\right| = \left|\frac{mZ_1}{4\left\{\frac{Z_2}{m} + \frac{(1-m^2)}{4m}Z_1\right\}}\right| = \left|\frac{m\omega L}{4\left\{\frac{1}{m\omega C} - \frac{(1-m^2)}{4m}\omega L\right\}}\right|$$

$$= \frac{m\omega L}{4\left\{\frac{1}{m\omega C} - \frac{(1-m^2)}{4m}\omega L\right\}}$$

$$= \frac{m\omega L}{4\left\{\frac{1}{m\omega C} - \frac{\frac{f_c{}^2}{f_\infty{}^2}}{4m}\omega L\right\}}$$

$$\because \; f_\infty = \frac{f_c}{\sqrt{(1-m^2)}}$$

$$(1-m^2) = \frac{f_c{}^2}{f_\infty{}^2}$$

$$= \frac{m\omega L}{4\left\{\frac{4-\left(\frac{f_c{}^2}{f_\infty{}^2}\right)\omega^2 LC}{4m\omega C}\right\}}$$

$$= \frac{m^2\omega^2 LC}{4-\left(\frac{f_c{}^2}{f_\infty{}^2}\right)\omega^2 LC}$$

$$= \frac{m^2\pi^2 f^2 LC}{4-\left(\frac{f_c{}^2}{f_\infty{}^2}\right)\pi^2 f^2 LC}$$

$$\because f_c = \frac{1}{\pi\sqrt{LC}}$$

$$\pi^2 LC = \frac{1}{f_c{}^2}$$

$$= \frac{m^2\frac{f^2}{f_c{}^2}}{1-\left(\frac{f_c{}^2}{f_\infty{}^2}\right)\frac{f^2}{f_c{}^2}}$$

$$= \frac{m^2\frac{f^2}{f_c{}^2}}{1-\left(\frac{f^2}{f_\infty{}^2}\right)}$$

so that for $f_c < f < f_\infty$

$$\alpha = 2\, cosh^{-1} \frac{m\frac{f}{f_c}}{\sqrt{1-\left(\frac{f^2}{{f_\infty}^2}\right)}}$$

and for $f_\infty < f$

$$\alpha = 2\, sinh^{-1} \frac{m\frac{f}{f_c}}{\sqrt{\left(\frac{f^2}{{f_\infty}^2}\right)-1}}$$

From the above expression α may be determined. Figure (3) shows a plot of a against $\frac{f}{f_c}$ for $m = 0.6$. It is observed that $f_\infty = 1.25$ times the cut off frequency f_c. The large increase in sharpness of cut off for the m derived section over the prototype is apparent. The constant β may be determined in the pass band from

$$\beta = 2\, sin^{-1} \frac{m\frac{f}{f_c}}{\sqrt{1-\left(\frac{f^2}{{f_\infty}^2}\right)}}$$

In the attenuation band, upto f_∞, β has the value π. Above f_∞ the value of β drops to zero, because the shunt arm becomes inductive above resonance. The phase shift of the m derived section is plotted as a function of $\frac{f}{f_c}$ in the figure (4).

The sharpness of cutoff increases for small values of m, the attenuation beyond the point of peak attenuation becomes smaller for small m. This emphasizes the necessity of supplementing the m derived section with a prototype section in series to raise attenuation for frequencies well remote from cut-off.

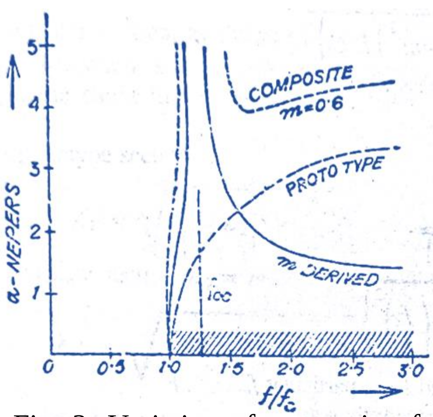


Fig. 3: Variation of attenuation for the prototype and m-derived section, and the composite result of the two in series

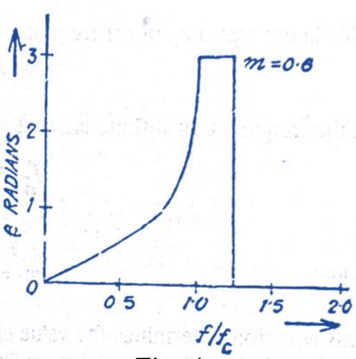


Fig. 4

References:

1. Hand Book of Electronics – Gupta & Kumar
2. Networks, Lines and Fields – John D. Ryder
3. Internet