2 Stress and infinitesimal strain

2.1 Problem definition

The engineering mechanics problem posed by underground mining is the prediction of the displacement field generated in the orebody and surrounding rock by any excavation and ore extraction processes. The rock in which excavation occurs is stressed by gravitational, tectonic and other forces, and methods exist for determining the ambient stresses at a mine site. Since the areal extent of any underground mine opening is always small relative to the Earth's surface area, it is possible to disregard the sphericity of the Earth. Mining can then be considered to take place in an infinite or semi-infinite space, which is subject to a definable initial state of stress.

An understanding of the notions of force, stress and strain is fundamental to a proper and coherent appreciation of the response of a rock mass to mining activity. It was demonstrated in Chapter 1 that excavating (or enlarging) any underground opening is mechanically equivalent to the application, or induction, of a set of forces distributed over the surfaces generated by excavation. Formation of the opening also induces a set of displacements at the excavation surface. From a knowledge of the induced surface forces and displacements, it is possible to determine the stresses and displacements generated at any interior point in the rock medium by the mining operation.

Illustration of the process of underground excavation in terms of a set of applied surface forces is not intended to suggest that body forces are not significant in the performance of rock in a mine structure. No body forces are induced in a rock mass by excavation activity, but the behaviour of an element of rock in the periphery of a mine excavation is determined by its ability to withstand the combined effect of body forces and internal, post-excavation surface forces. However, in many mining problems, body force components are relatively small compared with the internal surface forces, i.e. the stress components.

Some mine excavation design problems, such as those involving a jointed rock mass and low-stress environments, can be analysed in terms of block models and simple statics. In most deep mining environments, however, the rock mass behaves as a continuum, at least initially. Prediction of rock mass response to mining therefore requires a working understanding of the concepts of force, traction and stress, and displacement and strain. The following discussion of these issues follows the treatments by Love (1944) and Jaeger (1969).

In the discussion, the usual engineering mechanics convention is adopted, with tensile normal stresses considered positive, and the sense of positive shear stress on any surface determined by the sense of positive normal stress. The geomechanics convention for the sense of positive stresses will be introduced subsequently.

2.2 Force and stress

The concept of stress is used to describe the intensity of internal forces set up in a body under the influence of a set of applied surface forces. The idea is quantified by defining the state of stress at a point in a body in terms of the area intensity of forces

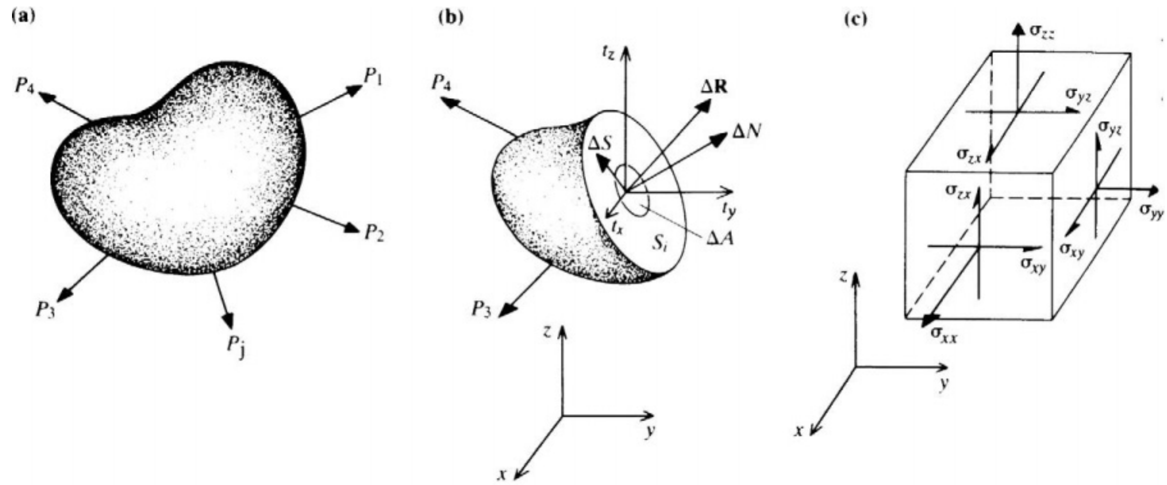


Figure 2.1 (a) A finite body subject to surface loading; (b) determination of the forces, and related quantities, operating on an internal surface; (c) specification of the state of stress at a point in terms of the traction components on the face of a cubic free body.

acting on the orthogonally oriented surfaces of an elementary free body centred on the point. If a Cartesian set of reference axes is used, the elementary free body is a cube whose surfaces are oriented with their outward normals parallel with the co-ordinate axes.

Figure 2.1a illustrates a finite body in equilibrium under a set of applied surface forces, P_j. To assess the state of loading over any interior surface, S_i, one could proceed by determining the load distribution over S_i required to maintain equilibrium of part of the body. Suppose, over an element of surface ΔA surrounding a point O, the required resultant force to maintain equilibrium is $\Delta\mathbf{R}$, as shown in Figure 2.1b. The magnitude of the **resultant stress** $\boldsymbol{\sigma}_r$ at O, or the **stress vector**, is then defined by

$$\boldsymbol{\sigma}_\mathbf{r} = \lim_{\Delta A \to 0} \frac{\Delta\mathbf{R}}{\Delta A}$$

If the vector components of $\Delta\mathbf{R}$ acting normally and tangentially to ΔA are ΔN, ΔS, the **normal stress** component, σ_n, and the resultant **shear stress** component, τ, at O are defined by

$$\sigma_n = \lim_{\Delta A \to 0} \frac{\Delta N}{\Delta A}, \quad \tau = \lim_{\Delta A \to 0} \frac{\Delta S}{\Delta A}$$

The stress vector, $\boldsymbol{\sigma}_r$, may be resolved into components t_x, t_y, t_z directed parallel to a set of reference axes x, y, z. The quantities t_x, t_y, t_z, shown in Figure 2.1b are called **traction components** acting on the surface at the point O. As with the stress vector, the normal stress, σ_n, and the resultant shear stress, τ, the traction components are expressed in units of force per unit area. A case of particular interest occurs when the outward normal to the elementary surface ΔA is oriented parallel to a co-ordinate axis, e.g. the x axis. The traction components acting on the surface whose normal is the x axis are then used to define three components of the state of stress at the point of interest,

$$\sigma_{xx} = t_x, \quad \sigma_{xy} = t_y, \quad \sigma_{xz} = t_z \tag{2.1}$$

In the doubly-subscripted notation for stress components, the first subscript indicates the direction of the outward normal to the surface, the second the sense of action of the stress component. Thus σ_{xz} denotes a stress component acting on a surface whose outward normal is the x axis, and which is directed parallel to the z axis. Similarly, for the other cases where the normals to elements of surfaces are oriented parallel to the y and z axes respectively, stress components on these surfaces are defined in terms of the respective traction components on the surfaces, i.e.

$$\sigma_{yx} = t_x, \quad \sigma_{yy} = t_y, \quad \sigma_{yz} = t_z \tag{2.2}$$

$$\sigma_{zx} = t_x, \quad \sigma_{zy} = t_y, \quad \sigma_{zz} = t_z \tag{2.3}$$

The senses of action of the stress components defined by these expressions are shown in Figure 2.1c, acting on the visible faces of the cubic free body.

It is convenient to write the nine stress components, defined by equations 2.1, 2.2, 2.3, in the form of a stress matrix $[\boldsymbol{\sigma}]$, defined by

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \tag{2.4}$$

The form of the stress matrix defined in equation 2.4 suggests that the state of stress at a point is defined by nine independent stress components. However, by consideration of moment equilibrium of the free body illustrated in Figure 2.1c, it is readily demonstrated that

$$\sigma_{xy} = \sigma_{yx}, \quad \sigma_{yz} = \sigma_{zy}, \quad \sigma_{zx} = \sigma_{xz}$$

Thus only six independent stress components are required to define completely the state of stress at a point. The stress matrix may then be written

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \tag{2.5}$$

2.3 Stress transformation

The choice of orientation of the reference axes in specifying a state of stress is arbitrary, and situations will arise in which a differently oriented set of reference axes may prove more convenient for the problem at hand. Figure 2.2 illustrates a set of old (x, y, z) axes and new (l, m, n) axes. The orientation of a particular axis, e.g. the l axis, relative to the original x, y, z axes may be defined by a row vector (l_x, l_y, l_z) of direction cosines. In this vector, l_x represents the projection on the x axis of a unit vector oriented parallel to the l axis, with similar definitions for l_y and l_z. Similarly, the orientations of the m and n axes relative to the original axes are defined by row vectors of direction cosines, (m_x, m_y, m_z) and (n_x, n_y, n_z) respectively. Also, the state of stress at a point may be expressed, relative to the l, m, n axes, by the stress matrix

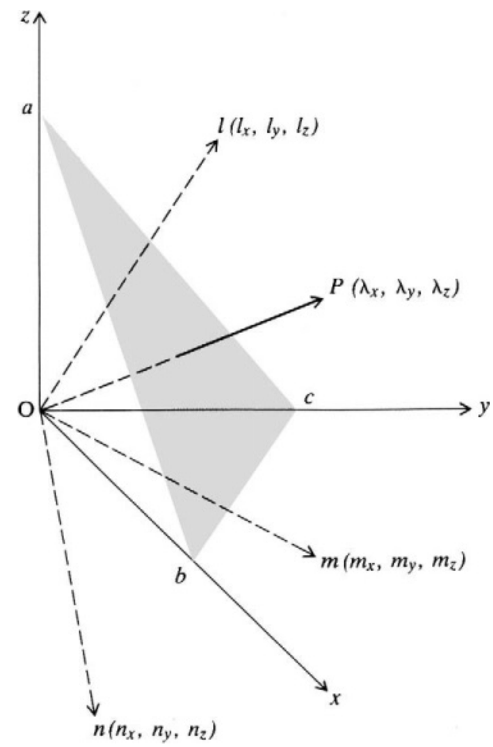


Figure 2.2 Free-body diagram for establishing the stress transformation equations, principal stresses and their orientations.

$[\boldsymbol{\sigma}^*]$, defined by

$$[\boldsymbol{\sigma}^*] = \begin{bmatrix} \sigma_{ll} & \sigma_{lm} & \sigma_{nl} \\ \sigma_{lm} & \sigma_{mm} & \sigma_{mn} \\ \sigma_{nl} & \sigma_{mn} & \sigma_{nn} \end{bmatrix}$$

The analytical requirement is to express the components of $[\boldsymbol{\sigma}^*]$ in terms of the components of $[\boldsymbol{\sigma}]$ and the direction cosines of the l, m, n axes relative to the x, y, z axes.

Figure 2.2 shows a tetrahedral free body, O*abc*, generated from the elementary cubic free body used to define the components of the stress matrix. The material removed by the cut *abc* has been replaced by the equilibrating force, of magnitude **t** per unit area, acting over *abc*. Suppose the outward normal OP to the surface *abc* is defined by a row vector of direction cosines $(\lambda_x, \lambda_y, \lambda_z)$. If the area of *abc* is A, the projections of *abc* on the planes whose normals are the x, y, z axes are given, respectively, by

$$\text{Area O}ac = A_x = A\lambda_x$$
$$\text{Area O}ab = A_y = A\lambda_y$$
$$\text{Area O}bc = A_z = A\lambda_z$$

Suppose the traction vector **t** has components t_x, t_y, t_z. Application of the equilibrium requirement for the x direction, for example, yields

$$t_x A - \sigma_{xx} A\lambda_x - \sigma_{xy} A\lambda_y - \sigma_{zx} A\lambda_z = 0 \tag{2.6}$$

or

$$t_x = \sigma_{xx}\lambda_x + \sigma_{xy}\lambda_y + \sigma_{zx}\lambda_z$$

Equation 2.6 represents an important relation between the traction component, the state of stress, and the orientation of a surface through the point. Developing the equilibrium equations, similar to equation 2.6, for the y and z directions, produces analogous expressions for t_y and t_z. The three equilibrium equations may then be written

$$\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \lambda_x \\ \lambda_y \\ \lambda_z \end{bmatrix} \tag{2.7}$$

or

$$[\mathbf{t}] = [\boldsymbol{\sigma}][\boldsymbol{\lambda}] \tag{2.8}$$

Proceeding in the same way for another set of co-ordinate axes l, m, n maintaining the same global orientation of the cutting surface to generate the tetrahedral free body, but expressing all traction and stress components relative to the l, m, n axes, yields the relations

$$\begin{bmatrix} t_l \\ t_m \\ t_n \end{bmatrix} = \begin{bmatrix} \sigma_{ll} & \sigma_{lm} & \sigma_{nl} \\ \sigma_{lm} & \sigma_{mm} & \sigma_{mn} \\ \sigma_{nl} & \sigma_{mn} & \sigma_{nn} \end{bmatrix} \begin{bmatrix} \lambda_l \\ \lambda_m \\ \lambda_n \end{bmatrix} \tag{2.9}$$

or

$$[\mathbf{t}^*] = [\boldsymbol{\sigma}^*][\boldsymbol{\lambda}^*] \tag{2.10}$$

In equations 2.8 and 2.10, $[\mathbf{t}]$, $[\mathbf{t}^*]$, $[\boldsymbol{\lambda}]$, $[\boldsymbol{\lambda}^*]$ are vectors, expressed relative to the x, y, z and l, m, n co-ordinate systems. They represent traction components acting on, and direction cosines of the outward normal to, a surface with fixed spatial orientation. From elementary vector analysis, a vector $[\mathbf{v}]$ is transformed from one set of orthogonal reference axes x, y, z to another set, l, m, n, by the transformation equation

$$\begin{bmatrix} v_l \\ v_m \\ v_n \end{bmatrix} = \begin{bmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

or

$$[\mathbf{v}^*] = [\mathbf{R}][\mathbf{v}] \tag{2.11}$$

In this expression, $[\mathbf{R}]$ is the rotation matrix, whose rows are seen to be formed from the row vectors of direction cosines of the new axes relative to the old axes.

As discussed by Jennings (1977), a unique property of the rotation matrix is that its inverse is equal to its transpose, i.e.

$$[\mathbf{R}]^{-1} = [\mathbf{R}]^{\mathrm{T}} \tag{2.12}$$

Returning now to the relations between $[\mathbf{t}]$ and $[\mathbf{t}^*]$, and $[\boldsymbol{\lambda}]$ and $[\boldsymbol{\lambda}^*]$, the results expressed in equations 2.11 and 2.12 indicate that

$$[\mathbf{t}^*] = [\mathbf{R}][\mathbf{t}]$$

or

$$[\mathbf{t}] = [\mathbf{R}]^{\mathrm{T}}[\mathbf{t}^*]$$

and

$$[\boldsymbol{\lambda}^*] = [\mathbf{R}][\boldsymbol{\lambda}]$$

or

$$[\boldsymbol{\lambda}] = [\mathbf{R}]^{\mathrm{T}}[\boldsymbol{\lambda}^*]$$

Then

$$\begin{aligned} [\mathbf{t}^*] &= [\mathbf{R}][\mathbf{t}] \\ &= [\mathbf{R}][\boldsymbol{\sigma}][\boldsymbol{\lambda}] \\ &= [\mathbf{R}][\boldsymbol{\sigma}][\mathbf{R}]^{\mathrm{T}}[\boldsymbol{\lambda}^*] \end{aligned}$$

but since

$$[\mathbf{t}^*] = [\boldsymbol{\sigma}^*][\boldsymbol{\lambda}^*]$$

then

$$[\boldsymbol{\sigma}^*] = [\mathbf{R}][\boldsymbol{\sigma}][\mathbf{R}]^{\mathrm{T}} \tag{2.13}$$

Equation 2.13 is the required **stress transformation equation**. It indicates that the state of stress at a point is transformed, under a rotation of axes, as a second-order tensor.

Equation 2.13 when written in expanded notation becomes

$$\begin{bmatrix} \sigma_{ll} & \sigma_{lm} & \sigma_{nl} \\ \sigma_{lm} & \sigma_{mm} & \sigma_{mn} \\ \sigma_{nl} & \sigma_{mn} & \sigma_{nn} \end{bmatrix} = \begin{bmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix}$$

Proceeding with the matrix multiplication on the right-hand side of this expression, in the usual way, produces explicit formulae for determining the stress components under a rotation of axes, given by

$$\sigma_{ll} = l_x^2\sigma_{xx} + l_y^2\sigma_{yy} + l_z^2\sigma_{zz} + 2(l_xl_y\sigma_{xy} + l_yl_z\sigma_{yz} + l_zl_x\sigma_{zx}) \tag{2.14}$$

$$\begin{aligned} \sigma_{lm} = {} & l_xm_x\sigma_{xx} + l_ym_y\sigma_{yy} + l_zm_z\sigma_{zz} + (l_xm_y + l_ym_x)\sigma_{xy} \\ & + (l_ym_z + l_zm_y)\sigma_{yz} + (l_zm_x + l_xm_z)\sigma_{zx} \end{aligned} \tag{2.15}$$

Expressions for the other four components of the stress matrix are readily obtained from these equations by cyclic permutation of the subscripts.

2.4 Principal stresses and stress invariants

The discussion above has shown that the state of stress at a point in a medium may be specified in terms of six components, whose magnitudes are related to arbitrarily selected orientations of the reference axes. In some rock masses, the existence of a particular fabric element, such as a foliation or a schistosity, might define a suitable direction for a reference axis. Such a feature might also determine a mode of deformation of the rock mass under load. However, in an isotropic rock mass, any choice of a set of reference axes is obviously arbitrary, and a non-arbitrary way is required for defining the state of stress at any point in the medium. This is achieved by determining **principal stresses** and related quantities which are invariant under any rotations of reference axes.

In section 2.2 it was shown that the resultant stress on any plane in a body could be expressed in terms of a normal component of stress, and two mutually orthogonal shear stress components. A **principal plane** is defined as one on which the shear stress components vanish, i.e. it is possible to select a particular orientation for a plane such that it is subject only to normal stress. The magnitude of the principal stress is that of the normal stress, while the normal to the principal plane defines the direction of the principal stress axis. Since there are, in any specification of a state of stress, three reference directions to be considered, there are three principal stress axes. There are thus three principal stresses and their orientations to be determined to define the state of stress at a point.

Suppose that in Figure 2.2, the cutting plane *abc* is oriented such that the resultant stress on the plane acts normal to it, and has a magnitude σ_p. If the vector $(\lambda_x, \lambda_y, \lambda_z)$ defines the outward normal to the plane, the traction components on *abc* are defined by

$$\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = \sigma_\mathrm{p} \begin{bmatrix} \lambda_x \\ \lambda_y \\ \lambda_z \end{bmatrix} \tag{2.16}$$

The traction components on the plane *abc* are also related, through equation 2.7, to the state of stress and the orientation of the plane. Subtracting equation 2.16 from equation 2.7 yields the equation

$$\begin{bmatrix} \sigma_{xx} - \sigma_\mathrm{p} & \sigma_{xy} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} - \sigma_\mathrm{p} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{yz} & \sigma_{zz} - \sigma_\mathrm{p} \end{bmatrix} \begin{bmatrix} \lambda_x \\ \lambda_y \\ \lambda_z \end{bmatrix} = [\mathbf{0}] \tag{2.17}$$

The matrix equation 2.17 represents a set of three simultaneous, homogeneous, linear equations in $\lambda_x, \lambda_y, \lambda_z$. The requirement for a non-trivial solution is that the determinant of the coefficient matrix in equation 2.17 must vanish. Expansion of the determinant yields a cubic equation in σ_p, given by

$$\sigma_\mathrm{p}^3 - I_1\sigma_\mathrm{p}^2 + I_2\sigma_\mathrm{p} - I_3 = 0 \tag{2.18}$$

In this equation, the quantities I_1, I_2 and I_3, are called the **first, second and third stress invariants**. They are defined by the expressions

$$
\begin{aligned}
I_1 &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \\
I_2 &= \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \left(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2\right) \\
I_3 &= \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - \left(\sigma_{xx}\sigma_{yx}^2 + \sigma_{yy}\sigma_{zx}^2 + \sigma_{zz}\sigma_{xy}^2\right)
\end{aligned}
$$

It is to be noted that since the quantities I_1, I_2, I_3 are invariant under a change of axes, any quantities derived from them are also invariants.

Solution of the characteristic equation 2.18 by some general method, such as a complex variable method, produces three real solutions for the principal stresses. These are denoted $\sigma_1, \sigma_2, \sigma_3$, in order of decreasing magnitude, and are identified respectively as the **major, intermediate** and **minor principal stresses**.

Each principal stress value is related to a principal stress axis, whose direction cosines can be obtained directly from equation 2.17 and a basic property of direction cosines. The dot product theorem of vector analysis yields, for any unit vector of direction cosines $(\lambda_x, \lambda_y, \lambda_z)$, the relation

$$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1 \tag{2.19}$$

Introduction of a particular principal stress value, e.g. σ_1, into equation 2.17, yields a set of simultaneous, homogeneous equations in $\lambda_{x1}, \lambda_{y1}, \lambda_{x1}$. These are the required direction cosines for the major principal stress axis. Solution of the set of equations for these quantities is possible only in terms of some arbitrary constant K, defined by

$$\frac{\lambda_{x1}}{A} = \frac{\lambda_{y1}}{B} = \frac{\lambda_{z1}}{C} = K$$

where

$$
\begin{aligned}
A &= \begin{vmatrix} \sigma_{yy} - \sigma_1 & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} - \sigma_1 \end{vmatrix} \\
B &= -\begin{vmatrix} \sigma_{xy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zz} - \sigma_1 \end{vmatrix} \\
C &= \begin{vmatrix} \sigma_{xy} & \sigma_{yy} - \sigma_1 \\ \sigma_{zx} & \sigma_{yz} \end{vmatrix}
\end{aligned}
\tag{2.20}
$$

Substituting for $\lambda_{x1}, \lambda_{y1}, \lambda_{z1}$ in equation 2.19, gives

$$
\begin{aligned}
\lambda_{x1} &= A/(A^2 + B^2 + C^2)^{1/2} \\
\lambda_{y1} &= B/(A^2 + B^2 + C^2)^{1/2} \\
\lambda_{z1} &= C/(A^2 + B^2 + C^2)^{1/2}
\end{aligned}
$$

Proceeding in a similar way, the vectors of direction cosines for the intermediate and minor principal stress axes, i.e. $(\lambda_{x2}, \lambda_{y2}, \lambda_{z2})$ and $(\lambda_{x3}, \lambda_{y3}, \lambda_{z3})$, are obtained from equations 2.20 by introducing the respective values of σ_2 and σ_3.

The procedure for calculating the principal stresses and the orientations of the principal stress axes is simply the determination of the eigenvalues of the stress matrix, and the eigenvector for each eigenvalue. Some simple checks can be performed to assess the correctness of solutions for principal stresses and their respective vectors of direction cosines. The condition of orthogonality of the principal stress axes requires that each of the three dot products of the vectors of direction cosines must vanish, i.e.

$$\lambda_{x1}\lambda_{x2} + \lambda_{y1}\lambda_{y2} + \lambda_{z1}\lambda_{z2} = 0$$

with a similar result for the (2,3) and (3,1) dot products. Invariance of the sum of the normal stresses requires that

$$\sigma_1 + \sigma_2 + \sigma_3 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

In the analysis of some types of behaviour in rock, it is usual to split the stress matrix into two components – a **spherical** or **hydrostatic component** $[\boldsymbol{\sigma}_{\mathrm{m}}]$, and a **deviatoric component** $[\boldsymbol{\sigma}_{\mathrm{d}}]$. The spherical stress matrix is defined by

$$[\boldsymbol{\sigma}_{\mathrm{m}}] = \sigma_{\mathrm{m}}[\mathbf{I}] = \begin{bmatrix} \sigma_{\mathrm{m}} & 0 & 0 \\ 0 & \sigma_{\mathrm{m}} & 0 \\ 0 & 0 & \sigma_{\mathrm{m}} \end{bmatrix}$$

where

$$\sigma_{\mathrm{m}} = I_1/3.$$

The deviator stress matrix is obtained from the stress matrix $[\sigma]$ and the spherical stress matrix, and is given by

$$[\boldsymbol{\sigma}_{\mathrm{d}}] = \begin{bmatrix} \sigma_{xx} - \sigma_{\mathrm{m}} & \sigma_{xy} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} - \sigma_{\mathrm{m}} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{yz} & \sigma_{zz} - \sigma_{\mathrm{m}} \end{bmatrix}$$

Principal deviator stresses S_1, S_2, S_3 can be established either from the deviator stress matrix, in the way described previously, or from the principal stresses and the hydrostatic stress, i.e.

$$S_1 = \sigma_1 - \sigma_{\mathrm{m}}, \text{ etc.}$$

where S_1 is the major principal deviator stress.

The principal directions of the deviator stress matrix $[\boldsymbol{\sigma}_{\mathrm{d}}]$ are the same as those of the stress matrix $[\boldsymbol{\sigma}]$.

2.5 Differential equations of static equilibrium

Problems in solid mechanics frequently involve description of the stress distribution in a body in static equilibrium under the combined action of surface and body forces. Determination of the stress distribution must take account of the requirement that the stress field maintains static equilibrium throughout the body. This condition requires satisfaction of the equations of static equilibrium for all differential elements of the body.

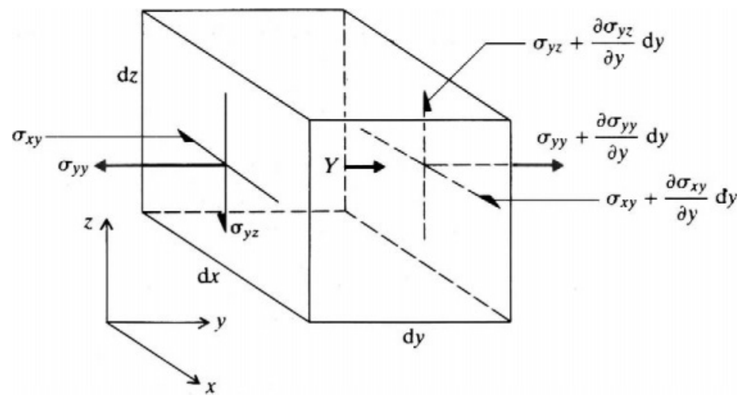


Figure 2.3 Free-body diagram for development of the differential equations of equilibrium.

Figure 2.3 shows a small element of a body, in which operate body force components with magnitudes X, Y, Z per unit volume, directed in the positive x, y, z co-ordinate directions. The stress distribution in the body is described in terms of a set of stress gradients, defined by $\partial\sigma_{xx}/\partial x$, $\partial\sigma_{xy}/\partial y$, etc. Considering the condition for force equilibrium of the element in the x direction yields the equation

$$\frac{\partial \sigma_{xx}}{\partial x} \cdot \mathrm{d}x \cdot \mathrm{d}y\,\mathrm{d}z + \frac{\partial \sigma_{xy}}{\partial y} \cdot \mathrm{d}y \cdot \mathrm{d}x\,\mathrm{d}z + \frac{\partial \sigma_{zx}}{\partial z} \cdot \mathrm{d}z \cdot \mathrm{d}x\,\mathrm{d}y + X\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0$$

Applying the same static equilibrium requirement to the y and z directions, and eliminating the term $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, yields the differential equations of equilibrium:

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + X &= 0 \\
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + Y &= 0 \\
\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + Z &= 0
\end{aligned} \tag{2.21}$$

These expressions indicate that the variations of stress components in a body under load are not mutually independent. They are always involved, in one form or another, in determining the state of stress in a body. A purely practical application of these equations is in checking the admissibility of any closed-form solution for the stress distribution in a body subject to particular applied loads. It is a straightforward matter to determine if the derivatives of expressions describing a particular stress distribution satisfy the equalities of equation 2.21.

2.6 Plane problems and biaxial stress

Many underground excavation design analyses involving openings where the length to cross section dimension ratio is high, are facilitated considerably by the relative simplicity of the excavation geometry. For example, an excavation such as a tunnel of uniform cross section along its length might be analysed by assuming that the stress distribution is identical in all planes perpendicular to the long axis of the excavation.

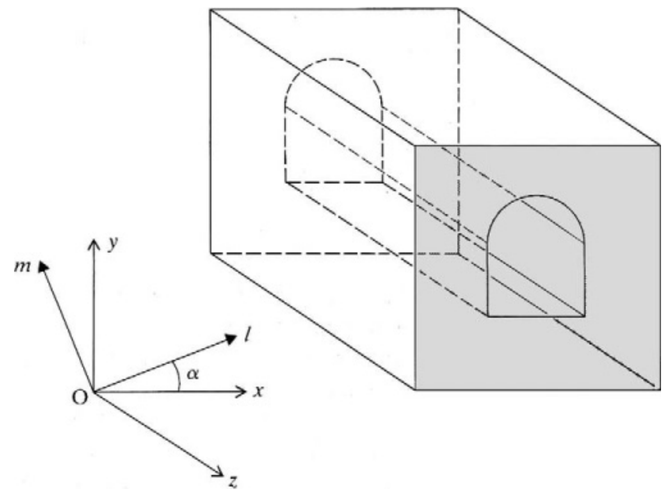


Figure 2.4 A long excavation, of uniform cross section, for which a contracted form of the stress transformation equations is appropriate.

Suppose a set of reference axes, x, y, z, is established for such a problem, with the long axis of the excavation parallel to the z axis, as shown in Figure 2.4. As shown above, the state of stress at any point in the medium is described by six stress components. For plane problems in the x, y plane, the six stress components are functions of (x, y) only. In some cases, it may be more convenient to express the state of stress relative to a different set of reference axes, such as the l, m, z axes shown in Figure 2.4. If the angle lOx is α, the direction cosines of the new reference axes relative to the old set are given by

$$\begin{aligned} l_x &= \cos\alpha, & l_y &= \sin\alpha, & l_z &= 0 \\ m_x &= -\sin\alpha, & m_y &= \cos\alpha, & m_z &= 0 \end{aligned}$$

Introducing these values into the general transformation equations, i.e. equations 2.14 and 2.15, yields

$$\begin{aligned} \sigma_{ll} &= \sigma_{xx}\cos^2\alpha + \sigma_{yy}\sin^2\alpha + 2\sigma_{xy}\sin\alpha\cos\alpha \\ \sigma_{mm} &= \sigma_{xx}\sin^2\alpha + \sigma_{yy}\cos^2\alpha - 2\sigma_{xy}\sin\alpha\cos\alpha \\ \sigma_{lm} &= \sigma_{xy}(\cos^2\alpha - \sin^2\alpha) - (\sigma_{xx} - \sigma_{yy})\sin\alpha\cos\alpha \\ \sigma_{mz} &= \sigma_{yz}\cos\alpha - \sigma_{zx}\sin\alpha \\ \sigma_{zl} &= \sigma_{yz}\sin\alpha + \sigma_{zx}\cos\alpha \end{aligned} \tag{2.22}$$

and the σ_{zz} component is clearly invariant under the transformation of axes. The set of equations 2.22 is observed to contain two distinct types of transformation: those defining $\sigma_{ll}, \sigma_{mm}, \sigma_{lm}$, which conform to second-order tensor transformation behaviour, and σ_{mz} and σ_{zl}, which are obtained by an apparent vector transformation. The latter behaviour in the transformation is due to the constancy of the orientation of the element of surface whose normal is the z axis. The rotation of the axes merely involves a transformation of the traction components on this surface.

For problems which can be analysed in terms of plane geometry, equations 2.22 indicate that the state of stress at any point can be defined in terms of the plane components of stress ($\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$) and the antiplane components ($\sigma_{zz}, \sigma_{yz}, \sigma_{zx}$). In the particular case where the z direction is a principal axis, the antiplane shear stress components vanish. The plane geometric problem can then be analysed in terms of the plane components of stress, since the σ_{zz}, component is frequently neglected. A state of biaxial (or two-dimensional) stress at any point in the medium is defined by three components, in this case $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$.

The stress transformation equations related to $\sigma_{ll}, \sigma_{mm}, \sigma_{lm}$ in equation 2.22, for the biaxial state of stress, may be recast in the form

$$\begin{aligned}
\sigma_{ll} &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \tfrac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\alpha + \sigma_{xy}\sin 2\alpha \\
\sigma_{mm} &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) - \tfrac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\alpha - \sigma_{xy}\sin 2\alpha \qquad (2.23) \\
\sigma_{lm} &= \sigma_{xy}\cos 2\alpha - \tfrac{1}{2}(\sigma_{xx} - \sigma_{yy})\sin 2\alpha
\end{aligned}$$

In establishing these equations, the x, y and l, m axes are taken to have the same sense of 'handedness', and the angle α is measured from the x to the l axis, in a sense that corresponds to the 'handedness' of the transformation. There is no inference of clockwise or anticlockwise rotation of axes in establishing these transformation equations. However, the way in which the order of the terms is specified in the equations, and related to the sense of measurement of the rotation angle α, should be examined closely.

Consider now the determination of the magnitudes and orientations of the plane principal stresses for a plane problem in the x, y plane. In this case, the $\sigma_{zz}, \sigma_{yz}, \sigma_{zx}$ stress components vanish, the third stress invariant vanishes, and the characteristic equation, 2.18, becomes

$$\sigma_p^2 - (\sigma_{xx} + \sigma_{yy})\sigma_p + \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 = 0$$

Solution of this quadratic equation yields the magnitudes of the plane principal stresses as

$$\sigma_{1,2} = \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) \pm \left[\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2\right]^{1/2} \qquad (2.24a)$$

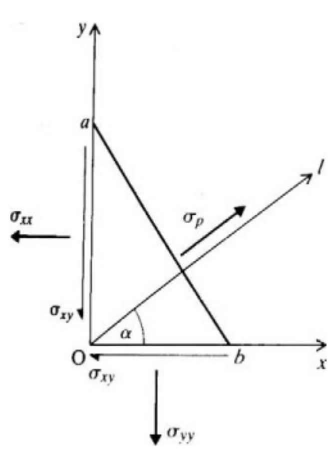


Figure 2.5 Problem geometry for determination of plane principal stresses and their orientations.

The orientations of the respective principal stress axes are obtained by establishing the direction of the outward normal to a plane which is free of shear stress. Suppose ab, shown in Figure 2.5, represents such a plane. The outward normal to ab is Ol, and therefore defines the direction of a principal stress, σ_p. Considering static equilibrium of the element aOb under forces operating in the x direction:

$$\sigma_p\, ab\cos\alpha - \sigma_{xx}\, ab\cos\alpha - \sigma_{xy}\, ab\sin\alpha = 0$$

or

$$\tan\alpha = \frac{\sigma_p - \sigma_{xx}}{\sigma_{xy}}$$

i.e.

$$\alpha = \tan^{-1}\frac{\sigma_p - \sigma_{xx}}{\sigma_{xy}} \qquad (2.24b)$$

Substitution of the magnitudes σ_1, σ_2, determined from equation 2.24a, in equation 2.24b yields the orientations α_1, α_2 of the principal stress axes relative to the positive direction of the x axis. Calculation of the orientations of the major and minor plane principal stresses in this way associates a principal stress axis unambiguously with a principal stress magnitude. This is not the case with other methods, which employ the last of equations 2.23 to determine the orientation of a principal stress axis.

It is to be noted that in specifying the state of stress in a body, there has been no reference to any mechanical properties of the body which is subject to applied load. The only concept invoked has been that of static equilibrium of all elements of the body.

2.7 Displacement and strain

Application of a set of forces to a body, or change in its temperature, changes the relative positions of points within it. The change in loading conditions from the initial state to the final state causes a displacement of each point relative to all other points. If the applied loads constitute a self-equilibrating set, the problem is to determine the equilibrium displacement field induced in the body by the loading. A particular difficulty is presented in the analysis of displacements for a loaded body where boundary conditions are specified completely in terms of surface tractions. In this case, unique determination of the absolute displacement field is impossible, since any set of rigid-body displacements can be superimposed on a particular solution, and still satisfy the equilibrium condition. Difficulties of this type are avoided in analysis by employing displacement gradients as the field variables. The related concept of strain is therefore introduced to make basically indeterminate problems tractable.

Figure 2.6 shows the original positions of two adjacent particles P(x, y, z) and Q$(x + dx, y + dy, z + dz)$ in a body. Under the action of a set of applied loads, P moves to the point P*$(x + u_x, y + u_y, z + u_z)$, and Q moves to the point Q*$(x + dx + u_x^*, y + dy + u_y^*, z + dz + u_z^*)$. If $u_x = u_x^*$, etc., the relative displacement between P

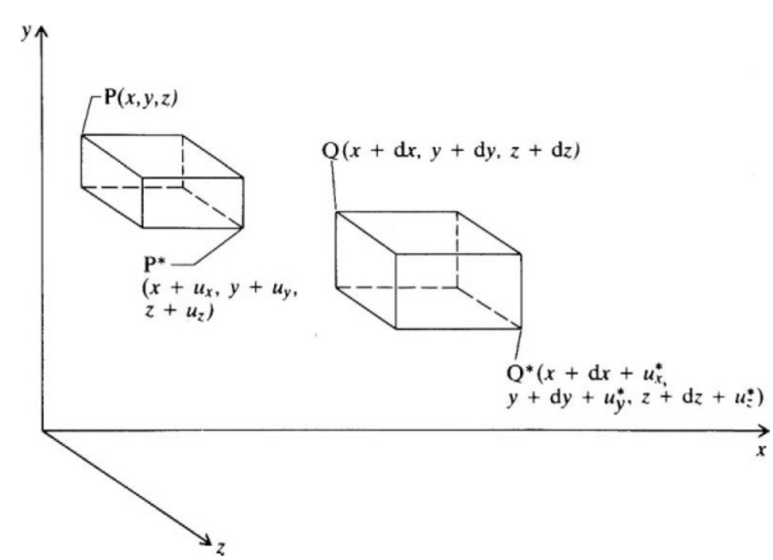


Figure 2.6 Initial and final positions of points P, Q, in a body subjected to strain.

and Q under the applied load is zero, i.e. the body has been subject to a rigid-body displacement. The problem of interest involves the case where $u_x \neq u_x^*$, etc. The line element joining P and Q then changes length in the process of load application, and the body is said to be in a state of strain.

In specifying the state of strain in a body, the objective is to describe the changes in the sizes and shapes of infinitesimal elements in the loaded medium. This is done by considering the displacement components (u_x, u_y, u_z) of a particle P, and (u_x^*, u_y^*, u_z^*) of the adjacent particle Q. Since

$$u_x^* = u_x + \mathrm{d}u_x, \quad \text{where } \mathrm{d}u_x = \frac{\partial u_x}{\partial x}\,\mathrm{d}x + \frac{\partial u_x}{\partial y}\,\mathrm{d}y + \frac{\partial u_x}{\partial z}\,\mathrm{d}z$$

and

$$u_y^* = u_y + \mathrm{d}u_y, \quad \text{where } \mathrm{d}u_y = \frac{\partial u_y}{\partial x}\,\mathrm{d}x + \frac{\partial u_y}{\partial y}\,\mathrm{d}y + \frac{\partial u_y}{\partial z}\,\mathrm{d}z$$

$$u_z^* = u_z + \mathrm{d}u_z, \quad \text{where } \mathrm{d}u_z = \frac{\partial u_z}{\partial x}\,\mathrm{d}x + \frac{\partial u_z}{\partial y}\,\mathrm{d}y + \frac{\partial u_z}{\partial z}\,\mathrm{d}z$$

the incremental displacements may be expressed by

$$\begin{bmatrix} \mathrm{d}u_x \\ \mathrm{d}u_y \\ \mathrm{d}u_z \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} & \dfrac{\partial u_x}{\partial y} & \dfrac{\partial u_x}{\partial z} \\ \dfrac{\partial u_y}{\partial x} & \dfrac{\partial u_y}{\partial y} & \dfrac{\partial u_y}{\partial z} \\ \dfrac{\partial u_z}{\partial x} & \dfrac{\partial u_z}{\partial y} & \dfrac{\partial u_z}{\partial z} \end{bmatrix} \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{bmatrix} \tag{2.25a}$$

or

$$[\mathrm{d}\boldsymbol{\delta}] = [\mathbf{D}][\mathrm{d}\mathbf{r}] \tag{2.25b}$$

In this expression, $[\mathrm{d}\mathbf{r}]$ represents the original length of the line element PQ, while $[\mathrm{d}\boldsymbol{\delta}]$ represents the relative displacement of the ends of the line element in deforming from the unstrained to the strained state.

The infinitesimal relative displacement defined by equation 2.25 can arise from both deformation of the element of which PQ is the diagonal, and a rigid-body rotation of the element. The need is to define explicitly the quantities related to deformation of the body. Figure 2.7 shows the projection of the element, with diagonal PQ, on to the yz plane, and subject to a rigid body rotation Ω_x about the x axis. Since the side dimensions of the element are $\mathrm{d}y$ and $\mathrm{d}z$, the relative displacement components of Q relative to P are

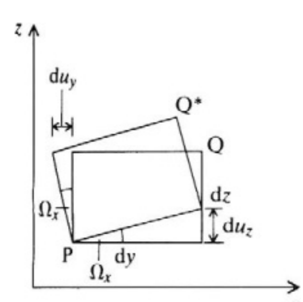


Figure 2.7 Rigid-body rotation of an element producing component displacements of adjacent points.

$$\begin{aligned} \mathrm{d}u_y &= -\Omega_x\,\mathrm{d}z \\ \mathrm{d}u_z &= \Omega_x\,\mathrm{d}y \end{aligned} \tag{2.26}$$

Considering rigid-body rotations Ω_y and Ω_z about the y and z axes, the respective displacements are

$$\begin{aligned} \mathrm{d}u_z &= -\Omega_y\,\mathrm{d}x \\ \mathrm{d}u_x &= \Omega_y\,\mathrm{d}z \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} du_x &= -\Omega_z\, dy \\ du_y &= \Omega_z\, dx \end{aligned} \tag{2.28}$$

The total displacement due to the various rigid-body rotations is obtained by addition of equations 2.26, 2.27 and 2.28, i.e.

$$\begin{aligned} du_x &= -\Omega_z\, dy + \Omega_y\, dz \\ du_y &= \Omega_z\, dx - \Omega_x\, dz \\ du_z &= -\Omega_y\, dx + \Omega_x\, dy \end{aligned}$$

These equations may be written in the form

$$\begin{bmatrix} du_x \\ du_y \\ du_z \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_z & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \tag{2.29a}$$

or

$$[d\boldsymbol{\delta}'] = [\boldsymbol{\Omega}][d\mathbf{r}] \tag{2.29b}$$

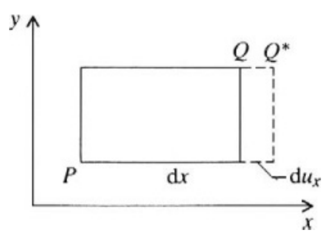


Figure 2.8 Displacement components produced by pure longitudinal strain.

The contribution of deformation to the relative displacement [dδ] is determined by considering elongation and distortion of the element. Figure 2.8 represents the elongation of the block in the x direction. The element of length dx is assumed to be homogeneously strained, in extension, and the normal strain component is therefore defined by

$$\varepsilon_{xx} = \frac{du_x}{dx}$$

Considering the y and z components of elongation of the element in a similar way, gives the components of relative displacement due to normal strain as

$$\begin{aligned} du_x &= \varepsilon_{xx}\, dx \\ du_y &= \varepsilon_{yy}\, dy \\ du_z &= \varepsilon_{zz}\, dz \end{aligned} \tag{2.30}$$

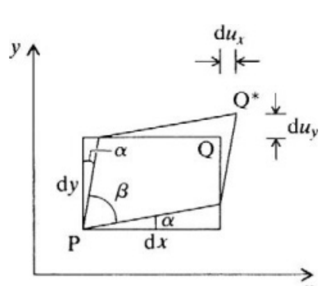


Figure 2.9 Displacement produced by pure shear strain.

The components of relative displacement arising from distortion of the element are derived by considering an element subject to various modes of pure shear strain. Figure 2.9 shows such an element strained in the x, y plane. Since the angle α is small, pure shear of the element results in the displacement components

$$\begin{aligned} du_x &= \alpha\, dy \\ du_y &= \alpha\, dx \end{aligned}$$

Since shear strain magnitude is defined by

$$\gamma_{xy} = \frac{\pi}{2} - \beta = 2\alpha$$

then

$$\begin{aligned} du_x &= \tfrac{1}{2}\gamma_{xy}\,dy \\ du_y &= \tfrac{1}{2}\gamma_{xy}\,dx \end{aligned} \tag{2.31}$$

Similarly, displacements due to pure shear of the element in the y, z and z, x planes are given by

$$\begin{aligned} du_y &= \tfrac{1}{2}\gamma_{yz}\,dz \\ du_z &= \tfrac{1}{2}\gamma_{yz}\,dy \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} du_z &= \tfrac{1}{2}\gamma_{zx}\,dx \\ du_x &= \tfrac{1}{2}\gamma_{zx}\,dz \end{aligned} \tag{2.33}$$

The total displacement components due to all modes of infinitesimal strain are obtained by addition of equations 2.30, 2.31, 2.32 and 2.33, i.e.

$$\begin{aligned} du_x &= \varepsilon_{xx}\,dx + \tfrac{1}{2}\gamma_{xy}\,dy + \tfrac{1}{2}\gamma_{zx}\,dz \\ du_y &= \tfrac{1}{2}\gamma_{xy}\,dx + \varepsilon_{yy}\,dy + \tfrac{1}{2}\gamma_{yz}\,dz \\ du_z &= \tfrac{1}{2}\gamma_{zx}\,dx + \tfrac{1}{2}\gamma_{yz}\,dy + \varepsilon_{zz}\,dz \end{aligned}$$

These equations may be written in the form

$$\begin{bmatrix} du_x \\ du_y \\ du_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \tfrac{1}{2}\gamma_{xy} & \tfrac{1}{2}\gamma_{zx} \\ \tfrac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \tfrac{1}{2}\gamma_{yz} \\ \tfrac{1}{2}\gamma_{zx} & \tfrac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \tag{2.34a}$$

or

$$[d\boldsymbol{\delta}''] = [\boldsymbol{\varepsilon}][d\mathbf{r}] \tag{2.34b}$$

where $[\boldsymbol{\varepsilon}]$ is the strain matrix.

Since

$$[d\boldsymbol{\delta}] = [d\boldsymbol{\delta}'] + [d\boldsymbol{\delta}'']$$

equations 2.25a, 2.29a and 2.34a yield

$$\begin{bmatrix} \dfrac{\partial u_x}{\partial x} & \dfrac{\partial u_x}{\partial y} & \dfrac{\partial u_x}{\partial z} \\ \dfrac{\partial u_y}{\partial x} & \dfrac{\partial u_y}{\partial y} & \dfrac{\partial u_y}{\partial z} \\ \dfrac{\partial u_z}{\partial x} & \dfrac{\partial u_z}{\partial y} & \dfrac{\partial u_z}{\partial z} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \tfrac{1}{2}\gamma_{xy} & \tfrac{1}{2}\gamma_{zx} \\ \tfrac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \tfrac{1}{2}\gamma_{yz} \\ \tfrac{1}{2}\gamma_{zx} & \tfrac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{bmatrix} + \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$$

Equating corresponding terms on the left-hand and right-hand sides of this equation, gives for the normal strain components

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{2.35}$$

and

$$\frac{\partial u_x}{\partial y} = \frac{1}{2}\gamma_{xy} - \Omega_z$$
$$\frac{\partial u_y}{\partial x} = \frac{1}{2}\gamma_{xy} + \Omega_z$$

Thus expressions for shear strain and rotation are given by

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \quad \Omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$$

and, similarly,

$$\begin{aligned} \gamma_{yz} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \quad \Omega_x = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) \\ \gamma_{zx} &= \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}, \quad \Omega_y = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) \end{aligned} \tag{2.36}$$

Equations 2.35 and 2.36 indicate that the state of strain at a point in a body is completely defined by six independent components, and that these are related simply to the displacement gradients at the point. The form of equation 2.34a indicates that a state of strain is specified by a second-order tensor.

2.8 Principal strains, strain transformation, volumetric strain and deviator strain

Since a state of strain is defined by a strain matrix or second-order tensor, determination of principal strains, and other manipulations of strain quantities, are completely analogous to the processes employed in relation to stress. Thus principal strains and principal strain directions are determined as the eigenvalues and associated eigenvectors of the strain matrix. Strain transformation under a rotation of axes is defined, analogously to equation 2.13, by

$$[\boldsymbol{\varepsilon}^*] = [\mathbf{R}][\boldsymbol{\varepsilon}][\mathbf{R}]^{\mathrm{T}}$$

where $[\boldsymbol{\varepsilon}]$ and $[\boldsymbol{\varepsilon}^*]$ are the strain matrices expressed relative to the old and new sets of co-ordinate axes.

The volumetric strain, Δ, is defined by

$$\Delta = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

The deviator strain matrix is defined in terms of the strain matrix and the volumetric strain by

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_{xx} - \Delta/3 & \gamma_{xy} & \gamma_{zx} \\ \gamma_{xy} & \varepsilon_{yy} - \Delta/3 & \gamma_{yz} \\ \gamma_{zx} & \gamma_{yz} & \varepsilon_{zz} - \Delta/3 \end{bmatrix}$$

Plane geometric problems, subject to biaxial strain in the xy plane, for example, are described in terms of three strain components, ε_{xx}, ε_{yy}, γ_{xy}.

2.9 Strain compatibility equations

Equations 2.35 and 2.36, which define the components of strain at a point, suggest that the strains are mutually independent. The requirement of physical continuity of the displacement field throughout a continuous body leads automatically to analytical relations between the displacement gradients, restricting the degree of independence of strains. A set of six identities can be established readily from equations 2.35 and 2.36. Three of these identities are of the form

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

and three are of the form

$$2\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

These expressions play a basic role in the development of analytical solutions to problems in deformable body mechanics.

2.10 Stress-strain relations

It was noted previously that an admissible solution to any problem in solid mechanics must satisfy both the differential equations of static equilibrium and the equations of strain compatibility. It will be recalled that in the development of analytical descriptions for the states of stress and strain at a point in a body, there was no reference to, nor exploitation of, any mechanical property of the solid. The way in which stress and strain are related in a material under load is described qualitatively by its **constitutive behaviour**. A variety of idealised constitutive models has been formulated for various engineering materials, which describe both the time-independent and time-dependent responses of the material to applied load. These models describe responses in terms of elasticity, plasticity, viscosity and creep, and combinations of these modes. For any constitutive model, stress and strain, or some derived quantities, such as stress and strain rates, are related through a set of constitutive equations. **Elasticity** represents the most common constitutive behaviour of engineering materials, including many rocks, and it forms a useful basis for the description of more complex behaviour.

In formulating constitutive equations, it is useful to construct column vectors from the elements of the stress and strain matrices, i.e. stress and strain vectors are defined by

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} \quad \text{and} \quad [\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

The most general statement of linear elastic constitutive behaviour is a generalised form of Hooke's Law, in which any strain component is a linear function of all the stress components, i.e.

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} \tag{2.37a}$$

or

$$[\boldsymbol{\varepsilon}] = [\mathbf{S}][\boldsymbol{\sigma}] \tag{2.37b}$$

Each of the elements S_{ij} of the matrix [**S**] is called a **compliance** or an **elastic modulus.** Although equation 2.37a suggests that there are 36 independent compliances, a reciprocal theorem, such as that due to Maxwell (1864), may be used to demonstrate that the compliance matrix is symmetric. The matrix therefore contains only 21 independent constants.

In some cases it is more convenient to apply equation 2.37 in inverse form, i.e.

$$[\boldsymbol{\sigma}] = [\mathbf{D}][\boldsymbol{\varepsilon}] \tag{2.38}$$

The matrix [**D**] is called the **elasticity matrix** or the matrix of **elastic stiffnesses**. For general anisotropic elasticity there are 21 independent stiffnesses.

Equation 2.37a indicates complete coupling between all stress and strain components. The existence of axes of elastic symmetry in a body de-couples some of the stress–strain relations, and reduces the number of independent constants required to define the material elasticity. In the case of **isotropic elasticity,** any arbitrarily oriented axis in the medium is an axis of elastic symmetry. Equation 2.37a, for isotropic elastic materials, reduces to

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} \tag{2.39}$$

The more common statements of Hooke's Law for isotropic elasticity are readily recovered from equation 2.39, i.e.

$$\varepsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})], \text{ etc.}$$
$$\gamma_{xy} = \frac{1}{G}\sigma_{xy}, \text{ etc.} \tag{2.40}$$

where

$$G = \frac{E}{2(1+\nu)}$$

The quantities E, G, and ν are **Young's modulus,** the **modulus of rigidity** (or **shear modulus**) and **Poisson's ratio**. Isotropic elasticity is a two-constant theory, so that determination of any two of the elastic constants characterises completely the elasticity of an isotropic medium.

The inverse form of the stress–strain equation 2.39, for isotropic elasticity, is given by

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \nu/(1-\nu) & \nu/(1-\nu) & 0 & 0 & 0 \\ \nu/(1-\nu) & 1 & \nu/(1-\nu) & 0 & 0 & 0 \\ \nu/(1-\nu) & \nu/(1-\nu) & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{(1-2\nu)}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{(1-2\nu)}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{(1-2\nu)}{2(1-\nu)} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} \tag{2.41}$$

The inverse forms of equations 2.40, usually called Lamé's equations, are obtained from equation 2.41, i.e.

$$\sigma_{xx} = \lambda\Delta + 2G\varepsilon_{xx}, \text{ etc.}$$
$$\sigma_{xy} = G\gamma_{xy}, \text{ etc.}$$

where λ is Lamé's constant, defined by

$$\lambda = \frac{2\nu G}{(1-2\nu)} = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

and Δ is the volumetric strain.

Transverse isotropic elasticity ranks second to isotropic elasticity in the degree of expression of elastic symmetry in the material behaviour. Media exhibiting transverse isotropy include artificially laminated materials and stratified rocks, such as shales. In the latter case, all lines lying in the plane of bedding are axes of elastic symmetry. The only other axis of elastic symmetry is the normal to the plane of isotropy. In Figure 2.10, illustrating a stratified rock mass, the plane of isotropy of the material

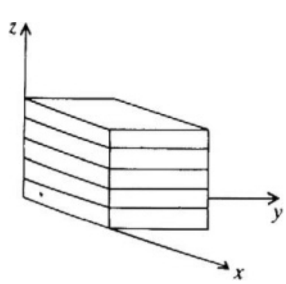


Figure 2.10 A transversely isotropic body for which the x, y plane is the plane of isotropy.

coincides with the x, y plane. The elastic constitutive equations for this material are given by

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \frac{1}{E_1} \begin{bmatrix} 1 & -\nu_1 & -\nu_2 & 0 & 0 & 0 \\ -\nu_1 & 1 & -\nu_2 & 0 & 0 & 0 \\ -\nu_2 & -\nu_2 & E_1/E_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & E_1/G_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & E_1/G_2 \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} \tag{2.42}$$

It appears from equation 2.42 that five independent elastic constants are required to characterise the elasticity of a transversely isotropic medium: E_1 and ν_1 define properties in the plane of isotropy, and E_2, ν_2, G_2 properties in a plane containing the normal to, and any line in, the plane of isotropy. Inversion of the compliance matrix in equation 2.42, and putting $E_1/E_2 = n$, $G_2/E_2 = m$, produces the elasticity matrix given by

$$[\mathbf{D}] = \frac{E_2}{(1+\nu_1)\left(1-\nu_1-2n\,\nu_2^2\right)} \begin{bmatrix} n\left(1-n\,\nu_2^2\right) & n\left(\nu_1+n\,\nu_2^2\right) & n^2\nu_2(1+\nu_1) & 0 & 0 & 0 \\ & n\left(1-n\,\nu_2^2\right) & n^2\nu_2(1+\nu_1) & 0 & 0 & 0 \\ & & \left(1-\nu_1^2\right) & 0 & 0 & 0 \\ & \text{symmetric} & & 0.5*n*\left(1-\nu_1-2n\,\nu_2^2\right) & 0 & 0 \\ & & & & m(1+\nu_1)*\left(1-\nu_1-2n\nu_2^2\right) & 0 \\ & & & & & m(1+\nu_1)*\left(1-\nu_1-2n\nu_2^2\right) \end{bmatrix}$$

Although it might be expected that the modulus ratios, n and m, and Poisson's ratios, ν_1 and ν_2, may be virtually independent, such is not the case. The requirement for positive definiteness of the elasticity matrix, needed to assure a stable continuum, restricts the range of possible elastic ratios. Gerrard (1977) has summarised the published experimental data on elastic constants for transversely isotropic rock materials and rock materials displaying other forms of elastic anisotropy, including orthotropy for which nine independent constants are required.

2.11 Cylindrical polar co-ordinates

A Cartesian co-ordinate system does not always constitute the most convenient system for specifying the state of stress and strain in a body, and problem geometry may suggest a more appropriate system. A cylindrical polar co-ordinate system is used frequently in the analysis of axisymmetric problems. Cartesian (x, y, z) and cylindrical polar (r, θ, z) co-ordinate systems are shown in Figure 2.11, together with an elementary free body in the polar system. To operate in the polar system, it is necessary to establish equations defining the co-ordinate transformation between Cartesian and polar co-ordinates, and a complete set of differential equations of equilibrium, strain displacement relations and strain compatibility equations.

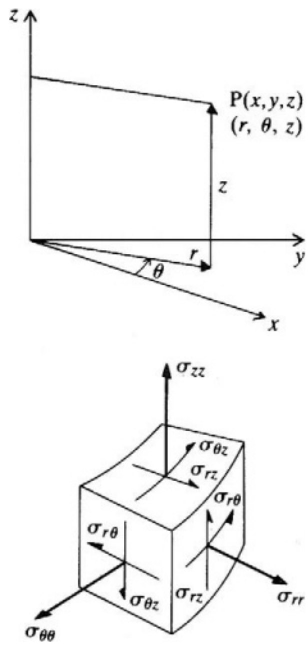


Figure 2.11 Cylindrical polar co-ordinate axes, and associated free-body diagram.

The co-ordinate transformation is defined by the equations.

$$r = (x^2 + y^2)^{1/2}$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$

and

$$x = r\cos\theta$$
$$y = r\sin\theta$$

If R, θ, Z are the polar components of body force, the differential equations of equilibrium, obtained by considering the condition for static equilibrium of the element shown in Figure 2.11, are

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + R = 0$$
$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + \theta = 0$$
$$\frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{zz}}{r} + Z = 0$$

For axisymmetric problems, the tangential shear stress components, $\sigma_{r\theta}$ and $\sigma_{\theta z}$, and the tangential component of body force, θ, vanish. The equilibrium equations reduce to

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + R = 0$$
$$\frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + Z = 0$$

For the particular case where r, θ, z are principal stress directions, i.e. the shear stress component σ_{rz} vanishes, the equations become

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + R = 0$$
$$\frac{\partial\sigma_{zz}}{\partial z} + Z = 0$$

Displacement components in the polar system are described by u_r, u_θ, u_z. The elements of the strain matrix are defined by

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$$
$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}$$
$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\gamma_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}$$
$$\gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}$$
$$\gamma_{r\theta} = \frac{1}{r}\left(-u_\theta + \frac{r\,\partial u_\theta}{\partial r} + \frac{\partial u_r}{\partial \theta}\right)$$

The volumetric strain is the sum of the normal strain components, i.e.

$$\Delta = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}$$

When the principal axes of strain coincide with the directions of the co-ordinate axes, i.e. the shear strain components vanish, the normal strains are defined by

$$\varepsilon_{rr} = \frac{\mathrm{d}u_r}{\mathrm{d}r}$$
$$\varepsilon_{\theta\theta} = \frac{u_r}{r}$$
$$\varepsilon_{zz} = \frac{\mathrm{d}u_z}{\mathrm{d}z}$$

The compatibility equations for strains are

$$\frac{\partial^2(r\gamma_{r\theta})}{\partial r\,\partial\theta} = r\frac{\partial^2(r\varepsilon_{\theta\theta})}{\partial r^2} - r\frac{\partial \varepsilon_{rr}}{\partial r} + \frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2}$$
$$\frac{\partial^2\gamma_{rz}}{\partial r\,\partial z} = \frac{\partial^2 \varepsilon_{rr}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial r^2}$$
$$\frac{\partial^2\gamma_{\theta z}}{\partial \theta\,\partial z} = \frac{\partial^2(r\varepsilon_{\theta\theta})}{\partial z^2} + \frac{1}{r}\frac{\partial^2 \varepsilon_{zz}}{\partial \theta^2} + \frac{\partial \varepsilon_{zz}}{\partial z} - \frac{\partial \gamma_{zr}}{\partial z}$$

The case where $\gamma_{r\theta} = \gamma_{\theta z} = \gamma_{rz} = 0$ yields only one compatibility equation, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\varepsilon_{\theta\theta}) = \varepsilon_{rr}$$

Stress components expressed relative to the Cartesian axes are transformed to the polar system using equations 2.22, with r and θ replacing l and m and θ replacing α. An analogous set of equations can be established for transformation of Cartesian strain components to the polar system.

2.12 Geomechanics convention for displacement, strain and stress

The convention used until now in the discussion of displacement, strain and stress has been the usual engineering mechanics one. Under this convention, force and displacement components are considered positive if directed in the positive directions of the co-ordinate axes. Extensile normal strains and tensile normal stresses are treated as positive. Finally, the sense of positive shear stress on a surface of the elementary free body is outward, if the outward normal to the surface is directed outward relative to the co-ordinate origin, and conversely. The sense of positive stress components, defined in this way, is illustrated in Figures 2.1c and 2.11, for Cartesian and polar co-ordinate systems. This convention has been followed in this introductory material since important notions such as traction retain their conceptual basis, and since practically significant numerical methods of stress analysis are usually developed employing it.

States of stress occurring naturally, and generated and sustained in a rock mass by excavation activity, are pervasively compressive. If the usual engineering mechanics convention for stresses were followed, all numerical manipulations related to stress and strain in rock would involve negative quantities. Although this presents no conceptual difficulties, convenience and accuracy in calculations are served by adopting the following convention for stress and strain analysis in rock mechanics:

(a) positive force and displacement components act in the positive directions of the co-ordinate axes;
(b) contractile normal strains are taken as positive;
(c) compressive normal stresses are taken as positive;
(d) the sense of positive shear stress on a surface is inward relative to the co-ordinate origin, if the inward normal to the surface acts inwards relative to the co-ordinate origin, and conversely.

The senses of positive stress components defined by this convention, for Cartesian and polar co-ordinate systems, and biaxial and triaxial states of stress, are shown in Figure 2.12. Some minor changes are required in some of the other general relations developed earlier, and these are now defined.

2.12.1 Stress-traction relations

If the outward normal to a surface has direction cosines $(\lambda_x, \lambda_y, \lambda_z)$, traction components are determined by

$$t_x = -(\sigma_{xx}\lambda_x + \sigma_{xy}\lambda_y + \sigma_{zx}\lambda_z), \text{ etc.}$$

2.12.2 Strain-displacement relations

Strain components are determined from displacement components using the expressions

$$\varepsilon_{xx} = -\frac{\partial u_x}{\partial x}$$

$$\gamma_{xy} = -\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right), \text{ etc.}$$

2.12.3 Differential equations of equilibrium

The change in the sense of positive stress components yields equations of the form

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} - X = 0, \text{ etc.}$$

All other relations, such as strain compatibility equations, transformation equations and stress invariants, are unaffected by the change in convention.

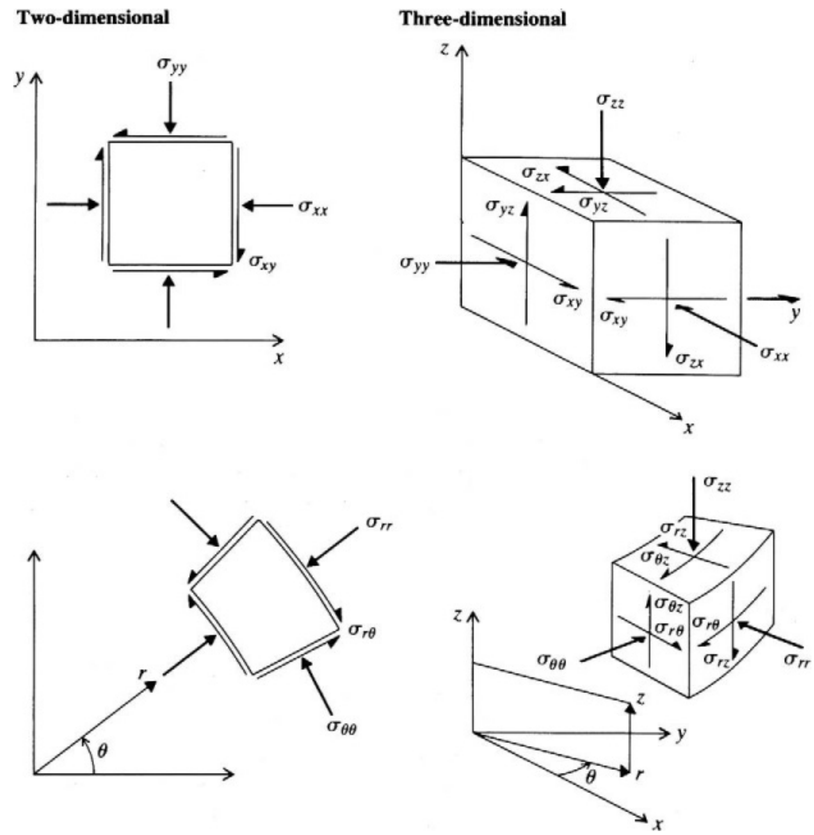


Figure 2.12 Two- and three-dimensional free bodies, for specification of the state of stress relative to Cartesian and polar co-ordinate axes, using the geomechanics convention for the sense of positive stresses.

2.13 Graphical representation of biaxial stress

Analytical procedures for plane problems subject to biaxial stress have been discussed above. Where equations or relations appropriate to the two-dimensional case have not been proposed explicitly, they can be established from the three-dimensional equations by deleting any terms or expressions related to the third co-ordinate direction. For example, for biaxial stress in the x, y plane, the differential equations of static equilibrium, in the geomechanics convention, reduce to

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} - X = 0$$
$$\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} - Y = 0$$

One aspect of biaxial stress that requires careful treatment is graphical representation of the state of stress at a point, using the **Mohr circle diagram**. In particular, the geomechanics convention for the sense of positive stresses introduces some subtle difficulties which must be overcome if the diagram is to provide correct determination of the sense of shear stress acting on a surface.

Correct construction of the Mohr circle diagram is illustrated in Figure 2.13. The state of stress in a small element abcd is specified, relative to the x, y co-ordinate

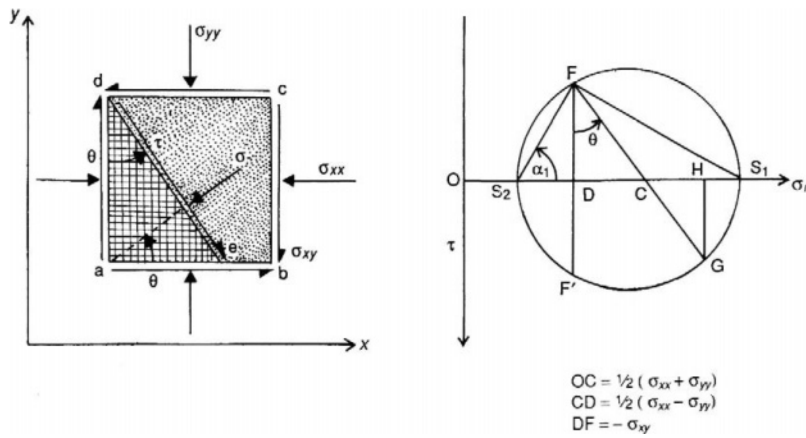


Figure 2.13 Construction of a Mohr circle diagram, appropriate to the geomechanics convention of stresses.

axes, by known values of σ_{xx}, σ_{yy}, σ_{xy}. A set of reference axes for the circle diagram construction is defined by directions σ_n and τ, with the sense of the positive τ axis directed downwards. If O is the origin of the $\sigma_n - \tau$ co-ordinate system, a set of quantities related to the stress components is calculated from

$$\mathrm{OC} = \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy})$$

$$\mathrm{CD} = \tfrac{1}{2}(\sigma_{xx} - \sigma_{yy})$$

$$\mathrm{DF} = -\sigma_{xy}$$

Points corresponding to C, D, F are plotted in the σ, τ plane as shown in Figure 2.13, using some convenient scale. In the circle diagram construction, if σ_{xy} is positive, the point F plots above the σ_n axis. Construction of the line FDF′ returns values of $\tau = \sigma_{xy}$ and $\sigma_n = \sigma_{xx}$ which are the shear and normal stress components acting on the surface cb of the element. Suppose the surface ed in Figure 2.13 is inclined at an angle θ to the negative direction of the y axis, or, alternatively, its outward normal is inclined at an angle θ to the x axis. In the circle diagram, the ray FG is constructed at an angle θ to FDF′, and the normal GH constructed. The scaled distances OH and HG then represent the normal and shear stress components on the plane ed.

A number of useful results can be obtained or verified using the circle diagram. For example, OS_1 and OS_2 represent the magnitudes of the major and minor principal stresses σ_1, σ_2. From the geometry of the circle diagram, they are given by

$$\begin{aligned}\sigma_{1,2} &= \mathrm{OC} \pm \mathrm{CF} \\ &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) \pm \left[\sigma_{xy}^2 + \tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2\right]^{1/2}\end{aligned}$$

confirming the solution given in equation 2.24a. The ray FS_1 defines the orientation of the major principal plane, so FS_2, normal to FS_1, represents the orientation of the major principal axis. If this axis is inclined at an angle α_1, to the x axis, the geometry of the circle diagram yields

$$\tan \alpha_1 = \frac{(OS_1 - OD)}{DF} = \frac{(\sigma_1 - \sigma_{xx})}{\sigma_{xy}}$$

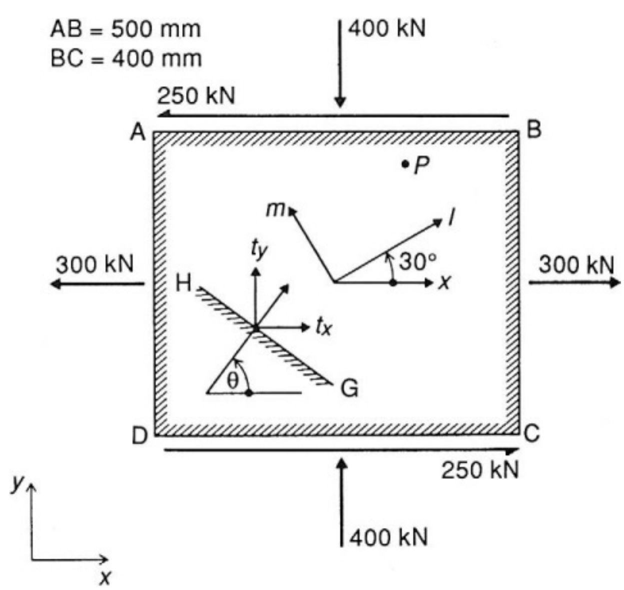
This expression is completely consistent with that for orientations of principal axes established analytically (equation 2.24b).

Problems

(The geomechanics convention for stress and strain is to be assumed in the following exercises.)

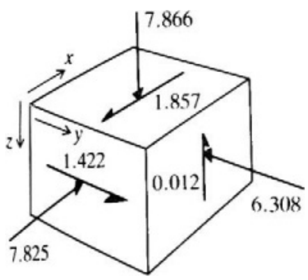
1 The rectangular plate shown in the figure below has the given loads uniformly distributed over the edges. The plate is 50 mm thick, AB is 500 mm and BC is 400 mm.

(a) Determine the shear forces which must operate on the edges BC, DA, to maintain the equilibrium of the plate.



(b) Relative to the x, y reference axes, determine the state of stress at any point P in the interior of the plate.
(c) For the l, m axes oriented as shown, determine the stress components $\sigma_{ll}, \sigma_{mm}, \sigma_{lm}$.
(d) Determine the magnitudes of the principal stresses, and the orientation of the major principal stress axis to the x axis.
(e) For the surface GH, whose outward normal is inclined at $\theta°$ to the x axis, determine expressions for the component tractions, t_x, t_y, operating on it as a

function of σ_{xx}, σ_{yy}, σ_{xy} and θ. Determine values of t_x, t_y for $\theta = 0°, 60°, 90°$, respectively. Determine the resultant stress on the plane for which $\theta = 60°$.



2 The unit free body shown in the figure (left) is subject to the stress components shown acting parallel to the given reference axes, on the visible faces of the cube.

(a) Complete the free-body diagram by inserting the required stress components, and specify the six stress components relative to the *x, y, z* axes.
(b) The *l, m, n* reference axes have direction cosines relative to the *x, y, z* axes defined by

$$(l_x, l_y, l_z) = (0.281, 0.597, 0.751)$$
$$(m_x, m_y, m_z) = (0.844, 0.219, -0.490)$$
$$(n_x, n_y, n_z) = (-0.457, 0.771, -0.442)$$

Write down the expressions relating σ_{mm}, σ_{nl} to the *x, y, z* components of stress and the direction cosines, and calculate their respective values.
(c) From the stress components established in (a) above, calculate the stress invariants, I_1, I_2, I_3, write down the characteristic equation for the stress matrix, and determine the principal stresses and their respective direction angles relative to the *x, y, z* axes.

Demonstrate that the principal stress directions define a mutually orthogonal set of axes.

3 A medium is subject to biaxial loading in plane strain. Relative to a set of x, y, co-ordinate axes, a load imposed at the co-ordinate origin induces stress components defined by

$$\sigma_{xx} = \frac{1}{r^2} - \frac{8y^2}{r^4} + \frac{8y^4}{r^6}$$
$$\sigma_{yy} = \frac{1}{r^2} + \frac{4y^2}{r^4} - \frac{8y^4}{r^6}$$
$$\sigma_{xy} = \frac{2xy}{r^4} - \frac{8xy^3}{r^6}$$

where $r^2 = x^2 + y^2$

Verify that the stress distribution described by these expressions satisfies the differential equations of equilibrium. Note that

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{x}{r^3} \text{ etc.}$$

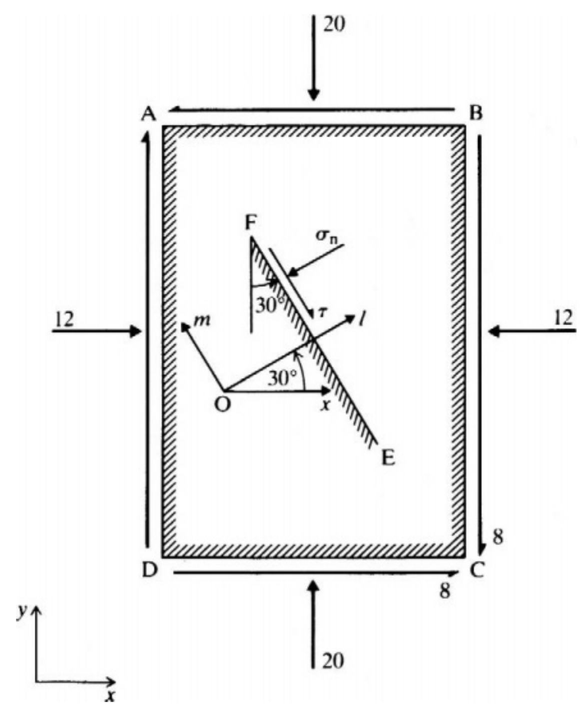
4 A medium is subject to plane strain loading by a perturbation at the origin of the x, y co-ordinate axes. The displacements induced by the loading are given by

$$u_x = \frac{1}{2G}\left[\frac{xy}{r^2} + C_1\right]$$
$$u_y = \frac{1}{2G}\left[\frac{y^2}{r^2} - (3 - 4v)\ell nr + C_2\right]$$

where C_1, C_2 are indefinite constants.

(a) Establish expressions for the normal and shear strain components, ε_{xx}, ε_{yy}, γ_{xy}.
(b) Verify that the expressions for the strains satisfy the strain compatibility equations.
(c) Using the stress–strain relations for isotropic elasticity, establish expressions for the stress components induced by the loading system.

5 The body shown in the figure below is subject to biaxial loading, with stress components given by $\sigma_{xx} = 12$, $\sigma_{yy} = 20$, $\sigma_{xy} = 8$.



(a) Construct the circle diagram representing this state of stress. Determine, from the diagram, the magnitudes of the principal stresses, and the inclination of the major principal stress axis relative to the x reference direction. Determine, from the diagram, the normal and shear stress components σ_n and τ on the plane EF oriented as shown.
(b) Noting that the outward normal, OL, to the surface EF is inclined at an angle of 30° to the x axis, use the stress transformation equations to determine the stress components σ_{ll} and σ_{lm}. Compare them with σ_n and τ determined in (a) above.
(c) Determine analytically the magnitudes and orientations of the plane principal stresses, and compare them with the values determined graphically in (a) above.