Computational Geology 12

Cramer's Rule and the Three-Point Problem

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Introduction

The three-point problem is one of the classic laboratory problems of the undergraduate geology curriculum. Given the elevation of three points on a geologic surface such as a formation contact, what is the attitude (strike and dip) of that surface? A similar question arises in hydrogeology. What is the gradient of the potentiometric surface given its elevation in three wells?

Aside from the fact that the three-point problem arises in real-world applications, geology instructors like the problem because it drives home the meaning of strike and dip.

The three-point problem is also a gateway to some useful mathematics. In this essay, I will discuss two solutions of the three-point problem using Cramer's Rule, an important technique for solving a small number of simultaneous equations. Cramer's Rule is one of the important methodologies of school algebra, but geology students generally do not see an application of simultaneous equations until advanced courses in geological data analysis or geophysics, and, in those courses, instructors want to approach higher-dimension problems using matrix algebra.
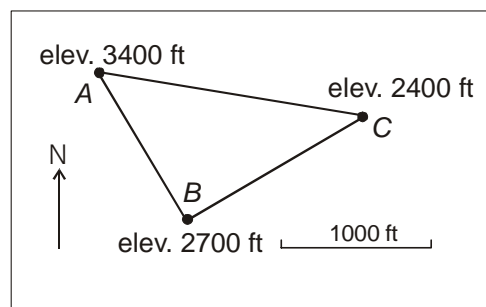
Figure 1. Map showing the location and elevation of three points. Modified slightly from Davis and Reynolds, 1996, Fig. G.7.

The Problem

Figure 1 shows a common presentation of the three-point problem. This example is very similar to the three-point problem discussed in a standard textbook in structural geology. The surface of interest is an unconformity. The elevation of the unconformity is known at three locations (*A*, *B*, and *C*). The horizontal scale is provided. We want to find the strike and dip of the unconformity.

Graphical Solution

   The standard approach is graphical. The elevation at *B* is between the elevations at *A* and *C*, so a contour passing through *B* (i.e., the 2700-ft contour) must cross the line segment *AC*. By the definition of strike, the direction of this contour is the strike of the unconformity surface. Thus the first step is to draw this contour (Fig. 2).

   We can locate the contour by dividing line segment *AC* into proportional parts according to the elevation differentials. Specifically, the unconformity surface drops 1000 ft between *A* and *C*; 700 ft between *A* and *B'*, and 300 ft between *B'* and *C*. Therefore, *B'* must be 70% of the distance from *A* to *C*. This locates *B'*. So, we draw the line segment *BB'* (Figure 2) and measure the azimuth of the strike with a protractor.
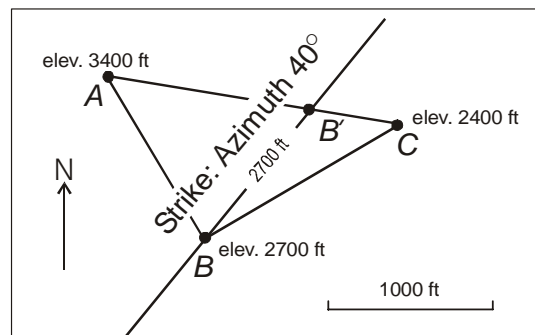
Figure 2. Map showing the location of the line of strike from data in Figure 1.

   The second step is to find the dip. To use a fully graphical way of doing this (Davis and Reynolds, 1996), draw a cross-section perpendicular to *BB'*, the line of strike (Figure 3). Then, using a vertical scale equal to the horizontal scale, lay out elevations on the cross section; project the locations of *A*, *B*, and *C* onto the cross-section at the appropriate elevations; and connect the dots. The resulting line segment shows the unconformity in cross-section. Because the cross-section is perpendicular to strike, the included angle is the true dip. So, we measure the angle with a protractor.
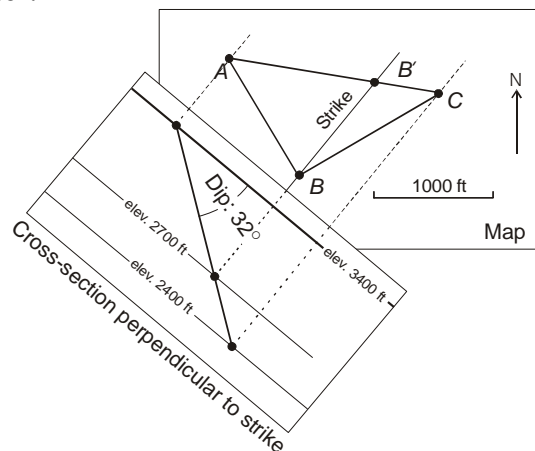
Figure 3. Map and cross-section showing true dip from data in Figure 1 (after Davis and Reynolds, 1996, Fig. G.7)

A slight modification. Sometimes it is easier to find drafting triangles and a ruler than it is to find a protractor. Not to worry, we can easily find the strike and dip by measuring distances and drawing parallels and perpendiculars with the triangles (Figure 4). First draw line segments AA' and CC' perpendicular to the line of strike. Second, draw right triangle BDC'' by drawing a vertical line (parallel to the north arrow) through B and a horizontal line segment (perpendicular to the north arrow) through C'. Then measure the distances BD, DC', AA', and CC' and find a calculator. The azimuth of strike (θ_{strike}) is

$$\theta_{strike} = \arctan\left(\frac{DC'}{BD}\right). \tag{1}$$

The angle of dip (θ_{dip}) is

$$\theta_{dip} = \arctan\left(\frac{h_A - h_{A'}}{AA'}\right) = \arctan\left(\frac{h_{C'} - h_C}{CC'}\right), \tag{2}$$

where h_A, $h_{A'}$, h_C, and $h_{C'}$ are the elevation at the locations identified by the subscripts.
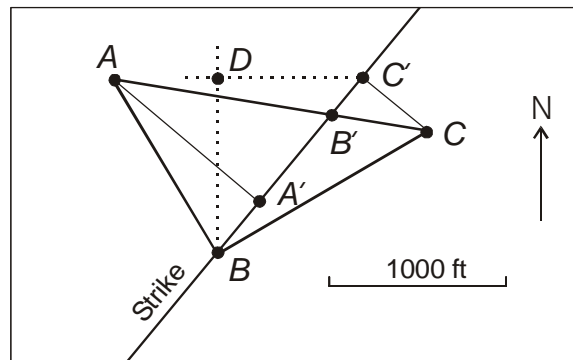

Figure 4. Map showing location of auxiliary lines to calculate strike and dip from arctangents.

   Limitations. Three-point problems make good graphical exercises. But graphical solutions take great care. If construction lines are off by only a slight angle, the error in the final answer can be substantial. Moreover, graphical solutions take time. How would you like to have to find the answer to 50 different three-point problems before leaving your desk?

Computational Solutions
   There are a number of ways of calculating the strike and dip from three-point data without measuring anything. These algorithms are based on analytical expressions and can be easily programmed. Such programs allow you to solve 50 or more three-point problems in the time that it takes you to enter the data.
   Computational solutions generally start with a different presentation of the problem. Figure 5 shows the same problem as Figure 1. The data are in Cartesian form; i.e., the x- y- and z-coordinates of the three points are given.
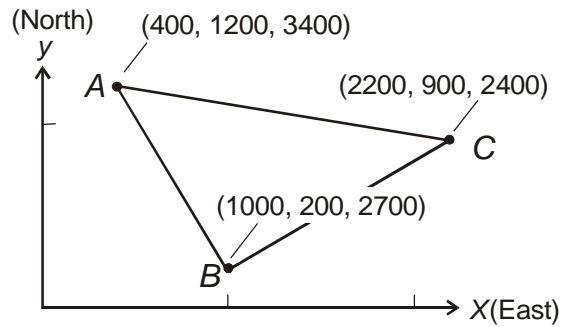
Figure 5. The same three-point problem in Cartesian form.

If the data are not given in Cartesian form originally, they can easily be converted to Cartesian form. For example, suppose we are given the elevation at all three points, the location of one of the points (let's say *A*), and the distance and direction from *A* to the other two points (lines *AB* and *AC*). From that information, we can calculate the *xy*-location of *B* and *C* by using the sine and cosine functions, and thereby have the *xyz*-location of all three points.

The rest of this essay addresses the problem posed in Figure 5. I will use the four-phase heuristic of Polya (1985) (see Computational Geology 6, Solving Problems, May 1999): (1) Understanding the problem, (2) Devising a plan, (3) Carrying out the plan, and (4) Looking back.

There are other computational solutions to the three-point problem. Three examples published in this journal are: Vacher (1989), DePaor (1991), and Sprenke (1992). All those examples, incidentally, include Cramer's Rule in the derivation.

Understanding the Problem

According to Polya's heuristic, understanding the problem involves visualizing the problem. "What is the target?" "What are the data?" "Make a sketch."

For seasoned geologists, Figure 6 might spring to mind. We are looking for the set of parallel, equally spaced contours of elevation (*z*) in the triangle defined by the three points as plotted on the map. The direction of the contours gives the strike. The spacing of the contours will lead to the dip.
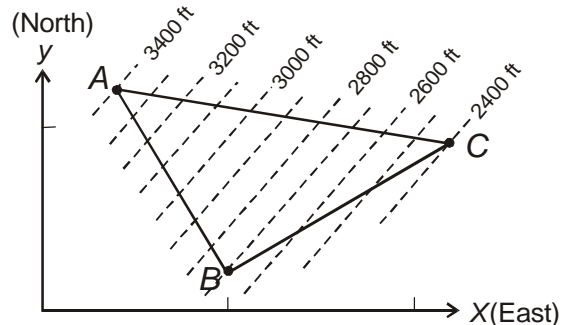
Figure 6. Contours of *z* on Figure 5.

That is fine for seasoned geologists who are at home with contours and have developed skill at envisioning objects three dimensionally. To them, Figure 6 captures the three-

dimensional essence of the problem. For others, it might be helpful to see an explicitly three-dimensional version of Figure 6. Figure 7, for example, shows the three points, the plane, and the contours on it suspended in space. The projection on the bottom surface (the *xy*-plane) shows the triangle of Figure 6. Also shown is the projection on the *xz*-plane, a side view. The contours would be horizontal lines on this projection.
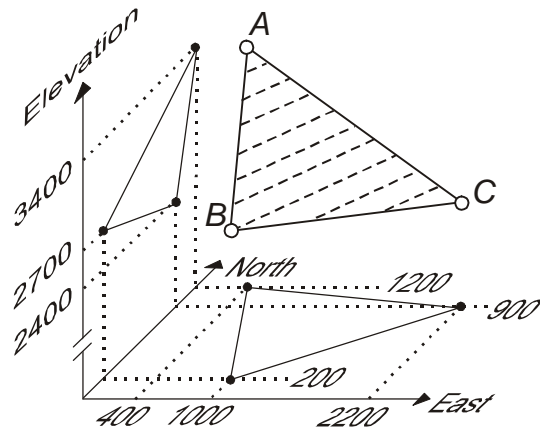
Figure 7. Three-dimensional context of the representation shown in Figure 6. (See also Isaaks and Srivastava, 1989, Fig. 11.3.)

Another of Polya's questions in understanding the problem is, "What is the condition?" We are clearly *assuming* that the surface within the triangle is a plane (which is why the contours are straight). The reason for making that assumption is simple: we do have no elevation data within the triangle. It is the simplest, most conservative assumption.

Devising a Plan, 1: Upgrading the Familiar Two-Point Problem
Under devising a plan, Polya (1985, p. xvi) suggested asking, "Have you seen (the problem) before? Or have you seen the same problem in a slightly different form? …. Do you know a related problem?" In other words, he recommended looking for analogies.
So, looking at Figure 7, is there an analogy to a familiar problem? Well, yes. We are asked to find the attitude of a plane joining (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3). This is simply the three-dimensional upgrade of the familiar two-point problem of school algebra: What is the slope of a line that connects (x_1, y_1) and (x_2, y_2)?
It is worth reviewing the two-dimensional, two-point problem before getting into the three-dimensional, three-point problem. The usual presentation of the two-point problem is: What is the *equation of the line* that passes through (x_1, y_1) and (x_2, y_2)?

The usual approach. By "equation of the line," the question usually is seeking the *slope-and-intercept form* of the equation:

$$y = y_0 + mx, \tag{3}$$

where y_0 is the intercept and m is the slope (Figure 8A). Given the equation of the line, one can immediately write down the slope (m) by inspection of the equation. More fundamentally – and

important to the three-dimensional generalization – one can get the slope by differentiating the equation of the line:

$$slope = \frac{dy}{dx} = \frac{d}{dx}\left(y_0 + mx\right) = m \,.$$ (4)
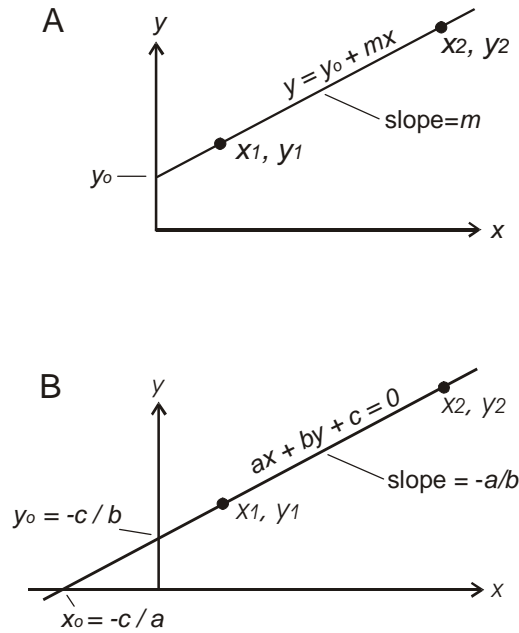
Figure 8. The two-dimensional two-point problem: equation of the straight line in slope-intercept form (A) and in linear-coefficients form (B).

Finding the equation of the line is straightforward. First, write the equation for the slope in terms of the data:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \,.$$ (5)

Then, consider an arbitrary point (x,y) that also lies on the line. Write the slope defined by (x_1, y_1) and (x, y):

$$m = \frac{y - y_1}{x - x_1} \,.$$ (6)

Set the two expressions for m equal to each other:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \,.$$ (7)

Solve for y in terms of x:

$$y = \left[y_1 - \frac{y_2 - y_1}{x_2 - x_1} x_1 \right] + \left(\frac{y_2 - y_1}{x_2 - x_1} \right) x \,. \tag{8}$$

By inspection, the term in the brackets is the y-intercept:

$$y_0 = y_1 - \frac{y_2 - y_1}{x_2 - x_1} x_1 \,. \tag{9}$$

The quantity in the parentheses in Equation 8 is the slope, which we already have (Equation 5). Equations 8, 5, and 9 are the answers to the two-dimensional, two-point problem.

  Another approach. There is another approach to the two-point problem that leads easily to three- and higher-dimensional problems. As shown in Figure 8B, the same line can be written with an equation in *linear-coefficients form*:

$$ax + by + c = 0 \,, \tag{10}$$

where a, b, and c are the linear coefficients. To convince yourself that Equation 10 is the equation of a line, rearrange Equation 10 to produce y as a function of x:

$$y = -\frac{c}{b} - \frac{a}{b} x \,. \tag{11}$$

This is the equation of the line in slope-and-intercept form. While you are at it, solve Equation 10 for x as a function of y:

$$x = -\frac{c}{a} - \frac{b}{a} y \,. \tag{12}$$

This is the equation of the line of x vs. y in slope-and-intercept form. From Equations 11 and 12, it is evident that ratios of the linear coefficients give the intercepts and slope (Figure 8B):

$$y_0 = -c/b \,, \tag{13}$$
$$x_0 = -c/a \,, \tag{14}$$

and
$$dy/dx = -a/b \,, \tag{15}$$
$$dx/dy = -b/a \,. \tag{16}$$

  The upgrade. Figure 9 shows the three-dimensional analog of Figure 8B: a plane passing through three non-collinear points. The equation of the plane is:
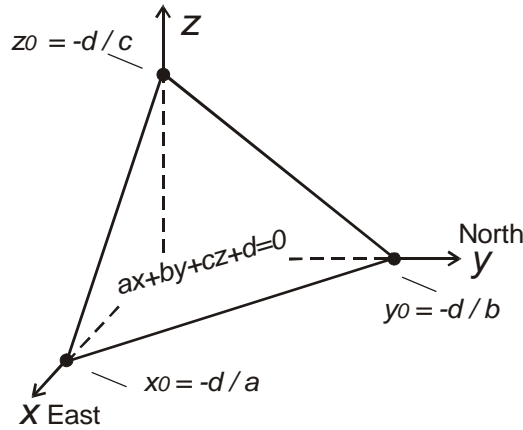
$$ax + by + cz + d = 0.$$ (17)

Letting two of the three independent variables (x, y and z) be zero immediately produces the three intercepts:

$$x_0 = -d/a,$$ (18)
$$y_0 = -d/b,$$ (19)
and $\quad z_0 = -d/c$ (20)

(compare with Equations 13 and 14).

Figure 9. The three-point problem: the equation of a plane in linear-coefficients form.

Letting one of the three independent variables at a time be zero produces the equations for the straight lines in each of the coordinate planes. Thus, in the xz-plane ($y = 0$),

$$ax + cz + d = 0.$$ (21)

The *slope* of the line in the xz-plane is

$$\partial z/\partial x = -a/c$$ (22)

from Equation 21 (compare with Equations 10 and 15 for the two-point problem).

The quantity $\partial z/\partial x$ in Equation 22 is the *partial derivative* of z with respect to x. This gives *the rate of change of z in the x-direction holding y constant* (hence, the slope of z vs. x as seen in the xz-plane).

Similarly, the equation of the line formed by the intersection of the plane with the yz-plane ($x = 0$) is

$$by + cz + d = 0,$$ (23)

and the slope of that line is

$$\partial z / \partial y = -b / c . \tag{24}$$

Finally, the intersection of the plane with xy-plane ($z = 0$) is given by

$$ax + by + d = 0 . \tag{25}$$

The strike of the plane. The xy-plane, of course, is horizontal, and so Equation 25 is the equation of a line of strike. The azimuth of strike (θ_{strike}) is the arctangent of the slope of this line plotted as x vs. y, or:

$$\theta_{strike} = \arctan(\partial x / \partial y) = \arctan(-b / a) \tag{26}$$

from Equation 16. The arctangent of the slope on a plot of y vs. x is $90°$ off strike and, therefore, gives the azimuth of the cross-section showing true dip ($\theta_{dipsection}$):

$$\theta_{dip\,section} = \arctan(\partial y / \partial x) = \arctan(-a / b) \tag{27}$$

from Equation 15. Thus both of these directions can be obtained easily from the equation of the plane (Equation 17).

The angle of dip. The partial derivative $\partial z/\partial x$ – the slope in the xz-cross-section – is the tangent of the *apparent dip* in the East direction. The partial derivative $\partial z/\partial y$ – the slope in the yz-cross-section – is the tangent of the *apparent dip* in the north direction. Together, the two partial derivatives can be used to find the *true dip* using the mathematical concept of gradient.

The *gradient* is the vector whose components are the partial derivatives along the coordinate axes [see Computational Geology 4, Mapping with vectors, January 1999, for vectors and their components]. In the case of our plane, the elevation, z, is a function of east (x) and north position (y): $z(x,y)$. Thus the vector gradient (∇z) is:

$$\nabla z = \frac{\partial z}{\partial x} \vec{i} + \frac{\partial z}{\partial y} \vec{j} . \tag{28}$$

The gradient (∇z) and its magnitude ($|\nabla z|$) are relevant to the three-point problem, because the gradient is the vector indicating the direction and magnitude of the largest slope of z at (x,y). From our experience as geologists walking around on dipslopes, we know that the slope (angle of climb) depends on direction: zero slope, if we walk the contour; maximum slope, if we walk perpendicular to the contours; something in between, if we walk switchbacks. The gradient of $z(x,y)$ is perpendicular to contours of z drawn on a map of xy-coordinates and points in the direction of larger z. The magnitude (length) of the vector is the slope.

Like all vectors, the magnitude of ∇z is found from the Pythagorean sum of its components:

$$|\nabla z| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \qquad . \qquad (29).$$

Because the $|\nabla z|$ is the maximum slope, the dip angle (θ_{dip}) is the arctangent of $|\nabla z|$. From Equations 29, 22, and 24, and some simplification, then,

$$\theta_{dip} = \arctan\left(\sqrt{\left(\frac{a^2 + b^2}{c^2}\right)}\right), \qquad (30)$$

which is easily calculated from the equation of the plane (Equation 17).

The direction of dip. Equation 27 produces the orientation of the dipsection, not the direction of dip. For example, if $\theta_{dipsection} = 50°$ (as in Figure 3), then the direction of dip could be either $-50°$ or $130°$. The two can be discriminated by the sign of the partial derivatives. For example, $\partial z/\partial x < 0$ and $\partial z/\partial y > 0$ (as in Figure 3) means that $130°$ is the correct direction of dip. For the other quadrants: the dip is to the southwest, if $\partial z/\partial x > 0$ and $\partial z/\partial y > 0$; to the northwest, if $\partial z/\partial x > 0$ and $\partial z/\partial y < 0$; and to the northeast, if $\partial z/\partial x < 0$ and $\partial z/\partial y < 0$.

If one of the partial derivatives is zero, then the dip is in one of the four cardinal directions. Thus, if $\partial z/\partial x = 0$, then the dip is south if $\partial z/\partial y > 0$, and north if $\partial z/\partial y < 0$. On the other hand, if $\partial z/\partial y = 0$, then the dip is west if $\partial z/\partial x > 0$, and east if $\partial z/\partial x < 0$.

Devising a Plan, 2: Cramer's Rule and the Two-Point Problem.
Determinants. Cramer's Rule uses determinants. Determinants are quantities that represent the sums and products of the numbers in a square array. Notationally, vertical lines enclose the determinant. For example, a 2×2 (second-order) determinant, $D_{n=2}$, contains four elements,

$$D_{n=2} = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}, \qquad (31)$$

where the subscripts of the α-elements refer to rows and columns (in that order) within the array. Similarly, a third-order determinant, $D_{n=3}$, contains nine elements:

$$D_{n=3} = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix}. \qquad (32)$$

The quantity represented by a second-order determinant is:

$$D_{n=2} = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}, \qquad (33)$$

which is obtained from the array by subtracting the product of the elements along the NE-SW diagonal from the product of the elements along the NW-SE diagonal. Thus, with the substitution instance of $\alpha_{11} = 1$, $\alpha_{12} = 2$, $\alpha_{21} = 3$, and $\alpha_{22} = 4$ (Figure 10A), the NW-SE diagonal (P1 [P for product]) is 4 and the NE-SW (P2) diagonal is 6; therefore, the quantity represented by the determinant is –2.

Similarly, the quantity represented by a third-order determinant is

$$D_{n=3} = \alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{31}\alpha_{12}\alpha_{23} + \alpha_{32}\alpha_{21}\alpha_{13} \\ - \alpha_{13}\alpha_{22}\alpha_{31} - \alpha_{33}\alpha_{12}\alpha_{21} + \alpha_{32}\alpha_{23}\alpha_{11} \tag{34}$$

which is obtained from the representation in Equation 32 by subtracting the sum of three NE-SW products from the sum of three NW-SE products as shown in Figure 10B. With the substitution instance shown in Figure 10B, Equation 34 works out to $45 + 84 + 96 - 105 - 72 - 48$, or 0.
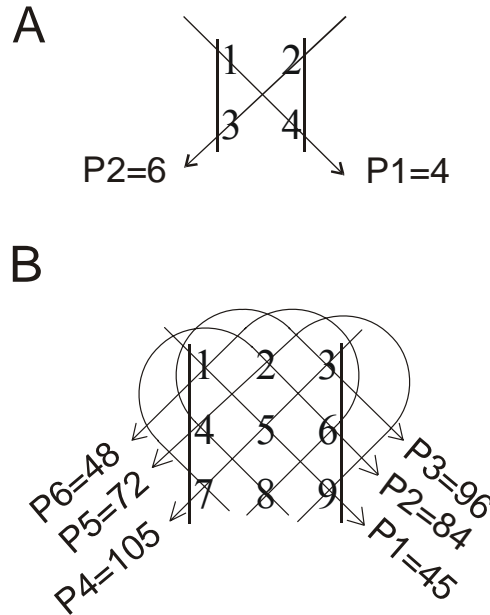
Figure 10. Lines of multiplication to evaluate a 2×2 determinant (A) and a 3×3 determinant (B).

A good exercise in school algebra is to use Equation 29 to show that

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} = \alpha_{11}\begin{vmatrix} \alpha_{22} & \alpha_{23} \\ \alpha_{32} & \alpha_{33} \end{vmatrix} - \alpha_{12}\begin{vmatrix} \alpha_{21} & \alpha_{23} \\ \alpha_{31} & \alpha_{33} \end{vmatrix} + \alpha_{13}\begin{vmatrix} \alpha_{21} & \alpha_{22} \\ \alpha_{31} & \alpha_{32} \end{vmatrix}. \tag{35}$$

Equation 35 is an example of expanding a determinant in terms of its cofactors. This is a procedure that is extremely useful, as we shall see. The general rule is that the determinant can be evaluated by summing the product of each α_{ij} across a row (or down a column) and the determinant remaining after blocking out the row and column containing α_{ij}. The signs in the

cofactor expansion are determined by the subscripts of α_{ij}: positive, if $i+j$ is even, and negative if $i+j$ is odd.

Cramer's Rule. Appendix 1 states Cramer's Rule in the full and general language of definitive textbooks in engineering mathematics. The statement is general in that it applies to n equations in n unknowns. It is definitive in that it gives the conditions for which the rule applies. Note that there are two parts (A and B) to the theorem in Appendix 1. Each part leads to a different method of finding the equation of the line. Together, the two methods clarify what the theorem means.

Appendix 2 states the equations in the general theorem (Appendix 1, part A) in more user-friendly language for systems of two and three simultaneous equations.

Application to the two-point problem, 1. Figure 8A shows two points on the line together with the equation of the line in slope-and-intercept form. The following two equations represent the fact that the two points (x_1, y_1) and (x_2, y_2) are on the line:

$$
\begin{aligned}
y_1 &= y_0 + mx_1 \\
y_2 &= y_0 + mx_2
\end{aligned}
\tag{36}
$$

They compose a system of two equations in two unknowns. The unknowns are m and y_0, the slope and intercept, respectively.

To solve Equations 36 for the two unknowns, first rearrange them so that they are in the same form as Equations A2.1 (Equations 1 of Appendix 2):

$$
\begin{aligned}
x_1 m + 1 y_0 &= y_1 \\
x_2 m + 1 y_0 &= y_2
\end{aligned}
\tag{37}
$$

Comparing Equations A2.1 and Equations 37 shows that the α-coefficients, α_{11}, α_{12}, α_{21} and α_{22} of Equations A2.1, match up with x_1, 1, x_2, and 1, respectively, of Equations 37. Similarly β_1 and β_2 are y_1 and y_2, respectively, of Equations 37. Finally, the unknowns $\underline{x}$ and $\underline{y}$ of Equations A2.1 are the m and y_0 of Equations 37. With those correspondences, the solution to the problem is

$$
m = \frac{\begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}}
\tag{38}
$$

and

$$
y_0 = \frac{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}}
\tag{39}
$$

from Equations A2.2-A2.5

So, given the two points, plug their coordinates into Equations 38 and 39, evaluate the determinants and their ratios, and, presto, you get the slope and intercept of the line passing through the points. Equations 38 and 39 are equivalent to Equations 5 and 9, respectively, which can be shown by evaluating the determinants (Fig. 10A).

Application to the two-point problem, 2. Figure 8B shows the two points and the equation of the line in linear-coefficients form. Consider the following system of three linear equations:

$$ax + by + c = 0$$
$$ax_1 + by_1 + c = 0 \ . \tag{40}$$
$$ax_2 + by_2 + c = 0$$

The first equation refers to a general point (x,y) and states that it is on the line. The second equation states that the particular point (x_1, y_1) is on the line. The third equation does the same for (x_2, y_2). The three linear coefficients, a, b and c, in Equations 40 are unknowns.

The procedure to find a, b and c is based on the first sentence of Part B of Appendix 1. First, we need to get Equations 40 into the form of Equations A2.6:

$$xa + yb + 1c = 0$$
$$x_1a + y_1b + 1c = 0 \ . \tag{41}$$
$$x_2a + y_2b + 1c = 0$$

Comparing Equations 41 to Equations A2.6, we see that the determinant of α-coefficients is:

$$D = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} . \tag{42}$$

Notice that all the β-terms in Equations 41 are zero. That is what it means to say that a set of linear equations is *homogeneous*. Therefore, from the first sentence of Part B of Appendix 1, it follows that

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 . \tag{43}$$

The algebra of the deduction is spelled out in Figure 11. The premises are (a) the first sentence of the second part of Cramer's Rule (Statement 1 of Fig. 11); (b) the fact that the β-terms are all zero (Statement 6); and (c) the fact that a nontrivial solution must exist (Statement 2). (In

connection with this last premise, consider what it would mean if a, b and c were all zero: Equations 40 would have no physical meaning.)

Symbols: $\wedge$ = "and"; $\vee$ = inclusive "or"; $\sim$ = "not"; $\rightarrow$ = "if ... then"

Propositions:
  p = "Equations 38 are homogeneous."
  q = "$D = 0$."
  r = "A nontrivial solution exists"

Derivation:

	Statement	Reason
1.	$[p \wedge \sim q] \rightarrow [\sim r]$	Given. Sentence 1 of Part B, Appendix 1.
2.	r	Given. The inclined plane exists. (We are trying to find its strike and dip!)
3.	$\sim [p \wedge \sim q]$	From 1 and 2, by *modus tollens* (negating the consequent).
4.	$\sim p \vee \sim [\sim q]$	From 3, by DeMorgan's Law.
5.	$\sim p \vee q$	From 4, by double negation.
6.	p	Given. β-terms are zero.
7.	$\sim [\sim p]$	From 6, by double negation.
8.	q	From 5 and 7, by disjunctive syllogism.

Figure 11. The logic producing Equation 41. For explanation of terms and symbols see CG10 - The Algebra of Deduction

Expanding Equation 43 using Equation 35 produces the equation of the line passing through (x_1, y_1) and (x_2, y_2):

$$ x \begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix} - y \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = 0 . \tag{44} $$

So, the linear coefficients of the line passing through (x_1, y_1) and (x_2, y_2) are:

$$ a = \begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix} , \tag{45} $$

$$ b = - \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} , \tag{46} $$

$$c = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}. \tag{47}$$

Substituting Equations 45 and 46 into Equation 15 produces the equation for slope (Equation 38). Substituting Equations 46 and 47 into Equation 13 produces the equation for the y-intercept (Equation 39).

Devising a Plan, 3: Cramer's Rule and the Three-Point Problem
    Option 1. The three-dimensional, three-point upgrade to the two-dimensional, two-point Equations 36 is:

$$\begin{aligned} z_1 &= m_x x_1 + m_y y_1 + z_0 \\ z_2 &= m_x x_2 + m_y y_2 + z_0 , \\ z_3 &= m_x x_3 + m_y y_3 + z_0 \end{aligned} \tag{48}$$

where the three unknowns are: m_x (the slope in the x-direction), m_y (the slope in the y-direction), and z_0 (the z-intercept). The coordinates of all the points (x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3), of course, are known. Then from comparison of Equations 48 with Equations A2.6, the determinant of α-coefficients is (from Equation A7):

$$D = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}. \tag{49}$$

The β-terms of Equations A2.6 are: $\beta_1 = z_1$, $\beta_2 = z_2$, and $\beta_3 = z_3$. Then, Equations A2.8, A2.9, and A2.10 are:

$$D_{m_x} = \begin{vmatrix} z_1 & y_1 & 1 \\ z_2 & y_2 & 1 \\ z_3 & y_3 & 1 \end{vmatrix}, \tag{50}$$

$$D_{m_y} = \begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}, \tag{51}$$

$$D_{z_0} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}, \tag{52}$$

respectively.

So to find the slopes and intercepts, evaluate the determinants of Equations 50-52 and calculate the following ratios:

$$m_x = D_{m_x} / D,$$ (53)

$$m_y = D_{m_y} / D,$$ (54)

$$z_0 = D_{z_0} / D.$$ (55)

These values provide the equation of the plane in slope-and-intercepts form.

Option 2. The three-dimensional, three-point upgrade to the two-dimensional, two-point Equations 40 is:

$$
\begin{aligned}
ax + by + cz + d &= 0 \\
ax_1 + by_1 + cz_1 + d &= 0 \\
ax_2 + by_2 + cz_2 + d &= 0 \\
ax_3 + by_3 + cz_3 + d &= 0
\end{aligned}
$$ (56)

The unknowns are a, b, c and d. Then, just as Equation 43 produces the equation of the line through (x_1, y_1) and (x_2, y_2),

$$
\begin{vmatrix}
x & y & z & 1 \\
x_1 & y_1 & z_1 & 1 \\
x_2 & y_2 & z_2 & 1 \\
x_3 & y_3 & z_3 & 1
\end{vmatrix} = 0
$$ (57)

produces the equation of the plane through (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) (compare with Sprenke, 1992, Equation 10). Expanding Equation 57 by its cofactors as in Equation 44 produces the linear coefficients:

$$
a = \begin{vmatrix}
y_1 & z_1 & 1 \\
y_2 & z_2 & 1 \\
y_3 & z_3 & 1
\end{vmatrix} = -\begin{vmatrix}
z_1 & y_1 & 1 \\
z_2 & y_2 & 1 \\
z_3 & y_3 & 1
\end{vmatrix},
$$ (58)

$$
b = -\begin{vmatrix}
x_1 & z_1 & 1 \\
x_2 & z_2 & 1 \\
x_3 & z_3 & 1
\end{vmatrix},
$$ (59)

$$
c = \begin{vmatrix}
x_1 & y_1 & 1 \\
x_2 & y_2 & 1 \\
x_3 & y_3 & 1
\end{vmatrix},
$$ (60)

and
$$d = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$
(61)

Equation 61 for the three-dimensional, three-point problem is analogous to Equation 47 for the two-dimensional, two-point problem. Similarly, compare Equations 58-60 to Equations 45 and 46.

The determinants associated with the linear-coefficients solution for the equation of the plane (Option 2, Equations 58-61) are closely related to those of the determinants associated with the slope-and-intercepts solution (Option 1, Equations 50-52). Thus:

$$a = -D_{m_x},$$
(62)

from Equations 58 and 50;

$$b = -D_{m_y},$$
(63)

from Equations 59 and 51;

$$c = D,$$
(64)

from Equations 60 and 49; and

$$d = -D_{z_0},$$
(65)

from Equations 61 and 52. These identities must hold because of Equation 20 and the following two relationships:

$$m_x = \partial z / \partial x = -a / c,$$
(66)

from Equation 22 and the meaning of partial derivative; and

$$m_y = \partial z / \partial y = -b / c,$$
(67)

from Equation 24 and the meaning of partial derivative.

Devising a Plan, 4: Conclusion
    Equations 26, 58 and 59 produce the azimuth of the strike of the plane. Equations 30, 58, 59 and 60 produce the angle of dip. Equations 22 and 24 produce the indicators that provide the direction dip.
    A plan, therefore, would consist of the following steps. First, use the coordinates of the three points to calculate the linear coefficients (a, b, c, and d) from Equations 58-61. Second,

calculate the partial derivatives (m_x and m_y) from Equations 22 and 24. Third, calculate the azimuth of strike from Equation 26. Fourth, calculate the angle of dip from Equation 30. Finally, determine the general direction of dip by inspection of m_x and m_y.

Carrying Out the Plan

As is typical of mathematical problem solving (Polya, 1957), devising the plan is the hard step. Carrying it out is a breeze by comparison.

The spreadsheet in Figure 12 carries out the plan. The boxed numbers in Part I are the data. The boxed numbers in Part IV are the answers (to two significant figures: strike = 40° and dip = 32° se). The unboxed numbers are intermediate steps. The determinants in Part II are created by references to cells in Part I. The determinants are evaluated in column H by a built-in function of the spreadsheet program. The slopes and intercept in Part III are from cell equations referring to the cells in column H of Part II. The strike and the angle of dip in Part IV are also from cell equations referring to cells in Column H of Part II. The result in Cell H31 is from a logic function referring to the slopes in Cells D30 and D31.

	B	C	D	E	F	G	H	I
3				THE THREE-POINT PROBLEM IN FIGURE 5				
4								
5	I. Coordinates of the three points							
6			x	y	z			
7		A	400	1200	3400			
8		B	1000	200	2700			
9		C	2200	900	2400			
10								
11	II. Liinear coefficients							
12			1200	3400	1			
13		a=	200	2700	1	=	790000	(Eqn 58)
14			900	2400	1			
15								
16			400	3400	1			
17		b=	1000	2700	1	=	-660000	(Eqn 59)
18			2200	2400	1			
19								
20			400	1200	1			
21		c=	1000	200	1	=	1620000	(Eqn 60)
22			2200	900	1			
23								
24			400	1200	3400			
25		d=	1000	200	2700	=	-5032000000	(Eqn 61)
26			2200	900	2400			
27								
28	III. Slopes and intercept					IV. Strike and dip		
29		z0=	3106.173	(Eqn 20)		Strike	39.87683446	(Eqn 26)
30		mx=	-0.487654	(Eqn 22)		Dip	32.43362326	(Eqn 30)
31		my=	0.407407	(Eqn 24)			se	

Figure 12. Spreadsheet to solve the problem in Figure 5. How many significant figures are appropriate for the answers?

From the results in Part II, Column H, the equation of the plane in linear-coefficients form is (three significant figures):

$$(7.90\times10^5)x - (6.60\times10^5)y + (1.62\times10^6)z - (5.03\times10^9) = 0 . \qquad (68)$$

From the results in Part III, Column D, the equation of the plane in slopes-and-intercept form is (three significant figures):

$$z = 3110 - 0.488x + 0.407y .$$ (69)

From the arctangents of the two slopes, the two apparent dips are: 26° to the east and 22° to the south.

Looking Back

Under "looking back," Polya (1957) asked: "Can you check your result?" There are various things we can do.

For starters, we can test the spreadsheet for cases with known answers. For example, consider the three points (0,0,100), (100,100,100), and (100,0,0). These are three corners of a cube 100 units on a side. The first two points form a diagonal across the upper face, the map view. The azimuth of this diagonal is 45°. The plane slices through this diagonal and passes through the southeastern corner of the bottom face of the cube. By the trigonometry of the 45° right triangle, the horizontal displacement is 50√2. Therefore, the slope of the plane is a drop of 100 units in a horizontal distance of 50√2, or $100/(50\sqrt{2}) = (\sqrt{2})$. The arctangent of √2 is 54.7°, which is the dip of the plane. The spreadsheet produces results of $m_x = -1$, $m_y = 1$, strike = 45°, and dip = 54.7° se.

Next, we can vary the three points around the corners of the cube so that the plane dips into different map quadrants. Thus, the three other examples are as follows. For (0,0,100), (100,100,100), and (0,100,0), the spreadsheet returns $m_x = 1$, $m_y = -1$, strike = 45° and dip = 54.7° nw. For (0,0,0), (0,100,100), and (100,0,100), the spreadsheet returns $m_x = 1$, $m_y = -1$, strike = 45°, and dip = 54.7° sw. Finally, for (0,100,100), (100,100,0), and (100, 0, 100), the spreadsheet returns $m_x = -1$, $m_y = -1$, strike = - 45°, and dip = 54.7° ne.

It is also a good idea to test the spreadsheet on examples that should have problems. Thus, (0,100,100), (100,0,100) and (100,100,100) define a horizontal plane which should not have a strike. The spreadsheet returns $m_x = 0$, $m_y = 0$, strike = #DIV/0!, and dip = 0°. An even more extreme reaction is produced by the three points (0,0,100), (50,50,100), and (100,100,100). For these three points, which lie on a straight line, the spreadsheet returns m_x = #DIV/0!, m_y = #DIV/0!, strike = #DIV/0!, and dip = #DIV/0!. The three-point problem does not exist for this case. One of the conditions of three-point problems is that the three points are *non-collinear*.

But what about the following three points: (0,0,100), (50,50,200), and (100,100,100)? These three points are non-collinear, but they lie on a straight line on a map. For them, the spreadsheet returns m_x = #DIV/0!, m_y = #DIV/0!, strike = 45°, and dip = #DIV/0!. How can one interpret this result? One way to see is to move one of the points slightly off the line on the map. Thus, for (0,0,100), (50,51,200), and (100,100,100), the spreadsheet returns m_x = -100, m_y = 100, strike = 45°, and dip = 89.59° se.

As another test, we can experimentally confirm that the dip calculated in the plane perpendicular to strike is larger than any apparent dip. This exercise makes use of our equations of planes and lines. Thus:

- Start with the equation for a line of strike. From Equations 68 and 25 and the values of the coefficients of Figure 12,

$$y = 1.19697x - 7621.212$$ (70)

(using excess significant figures for the purpose of comparisons).

- Find the equation of the line of dipsection through Point A. A line perpendicular to strike would have a slope of $-1/1.19697$ or -0.835443 from Equation 70. With this slope, the x- and y-coordinates of A (400 and 1200, respectively), and Equation 6, the line of dipsection through A is along

$$y = -0.835443x + 1534.177 .\qquad(71)$$

- Use Equation 71 to find another location on the dipsection. For example, if $x = 800$ ft, then from Equation 71, $y = 865.823$ ft, and so the point (800, 865.823) is also on the dipsection; call this point D. Calculate the horizontal distance between A and D using the Pythagorean Theorem; the answer is 521.224 ft. Use the xy-coordinates in Equation 69 to find the elevation of the unconformity at D; the answer (calculated with the original values of z_0, m_x and m_y from Figure 12) is 3068.79 ft. This means there is a drop of 3400-3068.79 or 331.21 ft in a distance of 521.224 ft. The arctangent of 331.21/521.224 is 32.43° to four significant figures – which agrees with the result in Figure 12.
- Select some other point (D') not on the line given by Equation 71; for example, let $x = 800$ and $y = 900$. Calculate the elevation of the unconformity at D' from Equation 69, the distance of D' from A by the Pythagorean Theorem, and finally, the apparent dip by the arctangent of the drop in elevation over the horizontal distance. The answer in this case is 32.40° to four significant figures. The result is less than the true dip.
- Repeat the last step until you are convinced that there is no other (x,y) that produces a dip larger than 32.43°.

Concluding Remarks

The key step in this solution of the three-point problem was finding the equation of the plane through the three points. After getting the equation of the plane, we found its strike and dip, using calculus for the latter. There are other things one can do once the equation of the plane is obtained. For example, one can find the elevation anywhere else in the triangle of Figure 1. This is the important problem of estimation in geostatistics (Isaaks and Srivastava, 1989), and it amounts to a three-dimensional version of linear interpolation.

To find the equation of the plane, we used simultaneous equations and solved them with Cramer's Rule. There are other ways of solving simultaneous equations. A particularly instructive way is to convert Equations 48 to a matrix equation and invert the matrix. Inverting the matrix illustrates another use of determinants.

Cramer's Rule and determinants are extremely useful in problem solving. We used Cramer's Rule (without much discussion) to solve one of the problems in "mapping with vectors" (CG-4, Equations 12). Determinants are priceless also in remembering how to calculate the vector cross product, which is used in solving problems in the physics of rotation and finding areas enclosed by vectors.

The key step in devising our plan to find the equation of the plane was the analogy with a similar problem in two dimensions. Finding analogies is commonly a fruitful strategy of problem solving. Polya (1957) gave more space to the topic of analogy than to any other single topic in his dictionary of heuristic.

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Appendices: Cramer's Rule

Appendix 1. General statement (rephrased slightly from Kreysig, 1993, p. 381, Theorem 2.)

(Part A) Let D be an n-rank determinant:

$$D = \begin{vmatrix} \alpha_{11} & \alpha_{12} & ... & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & ... & \alpha_{2n} \\ & .... & \\ \alpha_{n1} & \alpha_{n2} & ... & \alpha_{nn} \end{vmatrix}. \tag{A1.1}$$

If D is derived from the coefficients in a system of n linear equations

$$\begin{aligned} \alpha_{11}x_1 + \alpha_{12}x_2 + ... + \alpha_{1n}x_n &= \beta_1 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + ... + \alpha_{2n}x_n &= \beta_2 \\ & .... \\ \alpha_{n1}x_1 + \alpha_{n2}x_2 + ... + \alpha_{nn}x_n &= \beta_n \end{aligned} \tag{A1.2}$$

in the same number of unknowns x_1, x_2, …, x_n, AND if D is not zero, THEN the system of equations (A1.2) has precisely one solution. The solution is given by the formulas:

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad ..., \quad x_n = \frac{D_n}{D}, \tag{A1.3}$$

where D_n is the determinant obtained from D by replacing in D the nth column by the column with the entries β_1, β_2, …, β_n from A1.2

(Part B) Hence, if A1.2 is homogeneous AND $D \neq 0$, THEN A1.2 has only the trivial solution $\underline{x_1} = 0, \underline{x_2} = 0, \ldots, \underline{x_n} = 0$. If $D = 0$, then the homogeneous system also has nontrivial solutions.

Note on notation: The unknowns $\underline{x_1}, \underline{x_2}, \ldots, \underline{x_n}$ are underlined here to emphasize the fact that they are *unknowns*. They are <u>not</u> the same as the (non-underlined) quantities x_1, x_2, and x_3 denoting the *known x*-coordinates of points as discussed in the main text.

Appendix 2. Cramer's Rule Formulas for Two- and Three-Variable Problems

Two equations in two unknowns ($\underline{x}$ and $\underline{y}$).

Given:

$$\alpha_{11}\underline{x} + \alpha_{12}\underline{y} = \beta_1$$
$$\alpha_{21}\underline{x} + \alpha_{22}\underline{y} = \beta_2 \quad \text{.}$$

(A2.1)

Then,

$$D = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}.$$

(A2.2)

Also:

$$D_{\underline{x}} = \begin{vmatrix} \beta_1 & \alpha_{12} \\ \beta_2 & \alpha_{22} \end{vmatrix};$$

(A2.3)

$$D_{\underline{y}} = \begin{vmatrix} \alpha_{11} & \beta_1 \\ \alpha_{21} & \beta_2 \end{vmatrix}.$$

(A2.4)

Then,

$$\underline{x} = \frac{D_{\underline{x}}}{D}, \quad and \quad \underline{y} = \frac{D_{\underline{y}}}{D}.$$

(A2.5)

Three equations in three unknowns ($\underline{x}$, $\underline{y}$, and $\underline{z}$).

Given:

$$\alpha_{11}\underline{x} + \alpha_{12}\underline{y} + \alpha_{13}\underline{z} = \beta_1$$
$$\alpha_{21}\underline{x} + \alpha_{22}\underline{y} + \alpha_{23}\underline{z} = \beta_2 \quad \text{.}$$
$$\alpha_{31}\underline{x} + \alpha_{32}\underline{y} + \alpha_{33}\underline{z} = \beta_3$$

(A2.6)

Then,

$$D = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix}.$$

(A2.7)

Also:

$$D_{\underline{x}} = \begin{vmatrix} \beta_1 & \alpha_{12} & \alpha_{13} \\ \beta_2 & \alpha_{22} & \alpha_{23} \\ \beta_3 & \alpha_{32} & \alpha_{33} \end{vmatrix} ; \qquad\qquad\qquad (A2.8)$$

$$D_{\underline{y}} = \begin{vmatrix} \alpha_{11} & \beta_1 & \alpha_{13} \\ \alpha_{21} & \beta_2 & \alpha_{23} \\ \alpha_{31} & \beta_2 & \alpha_{33} \end{vmatrix} ; \qquad\qquad\qquad (A2.9)$$

$$D_{\underline{z}} = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \beta_1 \\ \alpha_{21} & \alpha_{22} & \beta_3 \\ \alpha_{31} & \alpha_{32} & \beta_3 \end{vmatrix} . \qquad\qquad\qquad (A2.10)$$

Then,

$$\underline{x} = \frac{D_{\underline{x}}}{D}, \quad \underline{y} = \frac{D_{\underline{y}}}{D}, \quad and \quad \underline{z} = \frac{D_{\underline{z}}}{D} . \qquad\qquad (A2.11)$$